

LINEAR OPERATORS ON L_p FOR $0 < p < 1$

BY

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ABSTRACT. If $0 < p < 1$ we classify completely the linear operators $T: L_p \rightarrow X$ where X is a p -convex symmetric quasi-Banach function space. We also show that if $T: L_p \rightarrow L_0$ is a nonzero linear operator, then for $p < q < 2$ there is a subspace Z of L_p , isomorphic to L_q , such that the restriction of T to Z is an isomorphism. On the other hand, we show that if $p < q < \infty$, the Lorentz space $L(p, q)$ is a quotient of L_p which contains no copy of l_p .

1. Introduction. The aim of this paper is to study and classify operators on the spaces $L_p(0, 1)$ for $0 < p < 1$ into other spaces of measurable functions. The underlying theme is the idea that operators on L_p cannot be "small" when $p < 1$. Historically the first result of this type was obtained by Day [4] in 1940 who showed that there is no nonzero operator of finite rank on L_p . Later, completing a partial result of Williamson [34], Pallaschke [26] and Turpin [31] showed that there is no compact endomorphism $T: L_p \rightarrow L_p$ other than zero. Recently the author [10] has shown that there is no nonzero compact operator on L_p with any range space. In fact, if T is a nonzero operator $T: L_p \rightarrow X$ (where X is any topological vector space) there is a subspace H of L_p isomorphic to l_2 such that $T|_H$ is an isomorphism.

It is quite possible that this last result can be improved quite substantially. To be precise we may ask the question for $0 < p < 1$:

QUESTION. Suppose $T: L_p \rightarrow X$ is a nonzero operator and $p < q < 2$. Does there exist a subspace Y of L_p such that $Y \cong L_q$ and $T|_Y$ is an isomorphism?

We do not know the answer to this or the weaker question with l_q replacing L_q . However in [11] we showed that if $T: L_p \rightarrow L_p$ is nonzero we can even obtain such a subspace $Y \cong L_p$. Of course, in general there is no hope of a result of this strength; consider the inclusion map $L_p \rightarrow L_0$ (that this does not preserve a copy of L_p is well known; it can be deduced easily from Example 9.9 below since $L_p \subset L(p, q) \subset L_0$ for $p < q < \infty$).

Our main result (Theorem 7.2) here will provide an affirmative answer to the question when $X = L_0$. Of course this also implies an affirmative answer for any space of functions densely embedded in L_0 , or more generally any space X such

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that the space of linear operators $\mathcal{L}(X, L_0)$ separates the points of X . Unfortunately, there are spaces X which are not too artificial such that $\mathcal{L}(X, L_0) = \{0\}$; an example was constructed by Christensen and Herer [2], but we also observe in §10 that $L_p(\Gamma)/H_p$ ($0 < p < 1$) is an example where Γ is the unit circle and H_p is the usual Hardy space.

Our other result related to the question above is a negative one. We show that a quotient of L_p need not contain l_p ($0 < p < 1$). We remark that a result of Turpin [32, p. 94] together with one of the author [15] shows that l_p must be finitely represented in any quotient of L_p . Indeed in [13] we defined a p -Banach space X to be p -trivial if $\mathcal{L}(L_p, X) = \{0\}$ and showed that this is an appropriate generalization of the Radon-Nikodym property. If X is not p -trivial, the l_p is finitely represented in X . The example in question here is the Lorentz space $L(p, q)$ where $p < q < \infty$.

Our main theorem is made possible by two other results. The first is Nikisin's theorem [21] that every linear operator $T \in \mathcal{L}(L_p, L_0)$ may be factored $L_p \rightarrow L(p, \infty) \rightarrow L_0$ where the second operator is a multiplication operator [here $L(p, \infty)$ is the weak space L_p]. The second is Theorem 6.1 that if $p < 1$, $L(p, \infty)$ is p -convex. We give in Theorem 6.4 a complete characterization of operators $T: L_p \rightarrow L[p, \infty]$ where $0 < p < 1$. Such an operator is of the form

$$Tf(t) = \sum_{n=1}^{\infty} a_n(t)f(\sigma_n t),$$

where $a_n: (0, 1) \rightarrow \mathbb{R}$ and $\sigma_n: (0, 1) \rightarrow (0, 1)$ are Borel maps and if

$$\alpha_x(B) = \sum_{n=1}^{\infty} m(|a_n| > x) \cap \sigma_n^{-1}B)$$

for B a Borel set then

$$\alpha_x(B) \leq Cx^p m(B), \quad B \in \mathfrak{B},$$

for some constant C . Conversely, any such $\{a_n\}, \{\sigma_n\}$ defines an operator $T: L_p \rightarrow L(p, \infty)$. This result is analogous to the result for endomorphisms of L_p given in [11].

We also initiate in §§8 and 9 a general study of p -Banach function spaces where $0 < p < 1$. There are certain differences from the theory of Banach function spaces, which give this study a distinctive flavour. One example is the fact noted above that if $p < 1$, $L(p, \infty)$ is p -convex; thus there are symmetric p -Banach function spaces which are strictly larger than L_p , while if $p = 1$, L_1 is the largest symmetric Banach function space and $L(1, \infty)$ is not locally convex. Lotz [18] and the author [14] have shown that if l_1 and L_1 embed in a Banach lattice with order-continuous norm then they embed as a sublattice. Here we give in §8 similar but slightly weaker results for l_p or L_p embedding in a p -Banach function space. The difficulty, as will be seen from the proofs, lies in the fact that we may not suppose a p -Banach function space densely embedded in L_p .

In §9 we study symmetric p -Banach function spaces ($0 < p < 1$) and introduce the class of totally symmetric p -Banach function spaces. Totally symmetric spaces lie between L_p and $L(p, \infty)$ and hence have no analogues for $p = 1$. We classify

completely operators from L_p into symmetric p -Banach function spaces. We show in particular that if X is a separable, σ -complete symmetric p -Banach function space, then X is isomorphic to a quotient of L_p if and only if X is totally symmetric.

We note at this point that the study of p -Banach lattices where $0 < p < 1$ is complicated by the fact that we cannot essentially reduce them to the study of function spaces; such a technique is available for Banach lattices (e.g., as in [14]). The possibility of such a reduction for p -Banach lattices with order-continuous quasinorm is closely related to Maharam's problem on the existence of a control measure for an order-continuous submeasure (cf. [2]).

We conclude with a note on the organization of the paper. §2 is purely to introduce notation. In §3, we give a treatment of Nikisin's theorem which is not essentially original. However we believe it may be useful to give a self-contained treatment. We also prove it in much stronger form than is required (the notion of type is not needed in this paper). §§4 and 5 develop some routine techniques. In §6 we prove our main representation theorem for operators from L_p into $L(p; \infty)$, and this leads in §7 to our main result on operators from L_p into L_0 . In §8 we study embeddings of l_p and L_p in p -Banach function spaces. In §9 we study symmetric p -Banach function spaces and operators as L_p . Finally, in §10 we give some results on translation-invariant operators on function spaces on compact groups and an example of a quotient of L_p which admits no nonzero operators into a p -Banach lattice with order-continuous quasinorm.

Our approach is not always economical. Results on $L(p, \infty)$ in §6 are special cases of results on symmetric spaces in §9 (Theorem 6.1 is a special case of Theorem 9.4; Corollary 6.5 is a special case of Theorem 9.6). However we felt there was an advantage in developing the theory for the main results rapidly before moving to a more general theory.

2. Prerequisites. A quasinorm on a real vector space X is a map $x \rightarrow \|x\|$ ($X \rightarrow \mathbf{R}$) such that

$$\|x\| > 0 \quad \text{if } x \neq 0, \quad (2.0.1)$$

$$\|tx\| = |t| \|x\|, \quad x \in X, t \in \mathbf{R}, \quad (2.0.2)$$

$$\|x + y\| \leq k(\|x\| + \|y\|), \quad x, y \in X, \quad (2.0.3)$$

where k is a constant independent of x and y . The best such constant k is called the modulus of concavity of the quasinorm. If $k = 1$, then the quasinorm is called a norm.

The sets $\{x: \|x\| < \varepsilon\}$ for $\varepsilon > 0$ form a base of neighborhoods for a Hausdorff vector topology on X . This topology is (locally) p -convex where $0 < p \leq 1$ if for some constant A and any $x_1, \dots, x_n \in X$,

$$\|x_1 + \dots + x_n\| \leq A(\|x_1\|^p + \dots + \|x_n\|^p)^{1/p}. \quad (2.0.4)$$

In this case we may endow X with an *equivalent* quasinorm,

$$\|x\|^* = \inf \left\{ \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p} : x_1 + \dots + x_n = x \right\}, \quad (2.0.5)$$

and then

$$\|x\| \geq \|x\|^* \geq A^{-1}\|x\|, \quad x \in X,$$

and $\|\cdot\|^*$ is p -subadditive, i.e.

$$\|x_1 + \cdots + x_n\|^* \leq \left(\sum_{i=1}^n \|x_i\|^{*p} \right)^{1/p}. \quad (2.0.6)$$

A theorem of Aoki and Rolewicz [28, p. 57] asserts that every quasinormed space is p -convex for some $p > 0$. A complete quasinormed space is called a *quasi-Banach space*, and if it is equipped with a p -subadditive quasinorm it is called a p -Banach space. If X and Y are quasi-Banach spaces then the space $\mathcal{L}(X, Y)$ of (bounded) linear operators $T: X \rightarrow Y$ is also a quasi-Banach space under the quasinorm

$$\|T\| = \sup(\|Tx\|: \|x\| \leq 1).$$

Suppose X is a vector lattice; then $\|\cdot\|$ is a lattice quasinorm on X if whenever $|x| \leq |y|$, $\|x\| \leq \|y\|$. A complete quasinormed lattice is called a *quasi-Banach lattice*. If X is a p -convex quasi-Banach lattice, then, as in Equation (2.0.5), X may be requasinormed with a p -subadditive *lattice* quasinorm; with such a quasinorm it is termed a p -Banach lattice.

Now let K be a compact metric space, and let $\mathfrak{B} = \mathfrak{B}(K)$ be the σ -algebra of Borel subsets on K . Suppose λ is a probability measure on K (i.e. a positive Borel measure of total mass one), with no atoms. Then $L_0(K) = L_0(K, \mathfrak{B}, \lambda)$ denotes the space of all Borel measurable real functions on K , where functions equal λ -almost everywhere are identified. This is an F -space (complete metric linear space) under the topology of convergence in measure. Then a *quasi-Banach function space* X on K is a subspace of $L_0(K)$ containing the simple functions such that if $g \in X$, $f \in L_0$ and $|f| \leq |g|$ (λ -a.e.) then $f \in X$, and equipped with a complete lattice quasinorm so that the inclusion map $X \hookrightarrow L_0$ is continuous. As usual if the associated quasinorm is p -subadditive then X is a p -Banach function space.

A quasi-Banach lattice X is σ -complete if whenever $\{x_n\}$ is a bounded increasing sequence of positive elements of X then $\{x_n\}$ has a supremum in X . If X is a σ -complete quasi-Banach function space then it may be requasinormed to have the *Fatou property*

$$\left\| \sup_n x_n \right\| = \sup_n \|x_n\| \quad (2.0.7)$$

for every such sequence; note that in this case $\sup_n x_n$ in X must coincide with the usual pointwise supremum in L_0 . We shall always assume the Fatou property is satisfied as in quasi-Banach function space.

Throughout this paper, K will denote an arbitrary, but fixed, compact metric space and we may suppress mention of the underlying probability measure space $(K, \mathfrak{B}, \lambda)$. Some examples of quasi-Banach function spaces on K are given by: (1) the spaces L_p ($0 < p < \infty$) of functions f such that

$$\|f\|_p = \left\{ \int_K |f(t)|^p d\lambda(t) \right\}^{1/p} < \infty,$$

(2) the weak L_p -spaces $L(p, \infty)$, $0 < p < \infty$ of functions f such that

$$\|f\|_{p,\infty} = \sup_{0 < x < \infty} x(\lambda(|f| > x))^{1/p} < \infty, \quad (2.0.8)$$

and (3) the Lorentz spaces $L(p, q)$, $0 < p < \infty$, $0 < q < \infty$ of functions f such that

$$\|f\|_{p,q} = \left\{ \frac{q}{p} \int_0^1 t^{q/p-1} f^*(t)^q dt \right\}^{1/q} < \infty, \quad (2.0.9)$$

where f^* is the decreasing rearrangement of $|f|$ on $(0, 1)$, i.e.

$$f^*(t) = \inf_{\lambda(E)=t} \sup_{s \in K-E} |f(s)|$$

(see Hunt [8], Saghar [29]).

Note that

$$\begin{aligned} \|f\|_{p,q}^q &= q/p \int_0^\infty t^{q/p-1} f^*(t)^q dt \\ &= \left[t^{q/p} f^*(t)^q \right]_0^\infty - \int_0^\infty t^{q/p} df^*(t)^q \\ &= q \int_0^\infty F(x)^{q/p} x^{q-1} dx, \end{aligned}$$

where $F(x) = \lambda(|f| > x)$ is the distribution of f (strictly, of course $1 - F$ is the distribution of f as a random variable). Hence

$$\|f\|_{p,q} = \left\{ q \int_0^\infty F(x)^{q/p} x^{q-1} dx \right\}^{1/q}. \quad (2.0.10)$$

A quasi-Banach function space is called *symmetric* if it is quasinormed such that $\|f\| = \|g\|$ whenever f and g have the same distribution. Note that each of the above examples is symmetric.

If $B \subset K$ is a Borel set, then 1_B denotes the indicator function of B and P_B , the natural projection of L_0 onto $L_0(B)$,

$$\begin{aligned} P_B f(s) &= f(s), \quad s \in B, \\ &= 0, \quad s \notin B. \end{aligned}$$

The space $\mathfrak{M}(K)$ is the space of signed Borel measures, so that $\mu(K) \approx C(K)^*$. If $\mu \in \mathfrak{M}(K)$, then $|\mu|$ is its total variation and if $0 < p < 1$, $|\mu|_p$ is its p -variation (cf. [11]). Then $\|\mu\| = |\mu|(K)$ and $\|\mu\|_p = (|\mu|_p(K))^{1/p}$; if $\|\mu\|_p < \infty$ then μ is purely atomic.

3. Nikisin's theorem. In this section we give a self-contained treatment of Nikisin's theorem on the factorization of operators into L_0 . In fact, Nikisin [21] proves a rather more general result for superlinear operators, but we shall specialize to the case which concerns us. We note that an earlier version of Nikisin's theorem is given in [19] and [20].

Suppose X is a quasi-Banach space. Then we say X is *type p* for $0 < p \leq 2$, if for some constant C and every $x_1, \dots, x_n \in X$,

$$\int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|^p dt \leq C \sum_{i=1}^n \|x_i\|^p,$$

where $r_1, \dots, r_n$ are the Rademacher functions on $(0, 1)$, $r_n(t) = \text{sgn}(\sin 2^n \pi t)$. If X is p -convex then it is also of type p ($0 < p \leq 1$).

THEOREM 3.1. *Suppose X is a quasi-Banach space of type p ($0 < p \leq 2$) and that $T: X \rightarrow L_0(K)$ is a linear operator. Then given $\varepsilon > 0$ there exists $E \in \mathfrak{B}$ with $\lambda(E) > 1 - \varepsilon$ and such that $P_E T \in \mathcal{L}(X, L(p, \infty))$ where*

$$\begin{aligned} P_E f(t) &= f(t), & t \in E, \\ &= 0, & t \notin E. \end{aligned}$$

PROOF. First we observe that since T is continuous, for every $\varepsilon > 0$ there exists $R(\varepsilon) < \infty$ such that if $\|x\| \leq 1$,

$$\lambda(|Tx| > R(\varepsilon)) \leq \varepsilon. \quad (3.1.1)$$

We now establish:

Given any $\varepsilon > 0$ and $x_1, \dots, x_n \in X$ there exists

$$E = E(x_1, \dots, x_n) \in \mathfrak{B} \text{ such that } \lambda(E) > 1 - \varepsilon \text{ and} \quad (3.1.2)$$

$$\left(\sum_{i=1}^n |Tx_i(s)|^2 \right)^{1/2} \leq A(\varepsilon) \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p} \quad (s \in E),$$

where

$$A(\varepsilon) = 4 \left(\frac{16C}{\varepsilon} \right)^{1/p} R\left(\frac{\varepsilon}{16}\right).$$

To prove (3.1.2) let us assume $\varepsilon > 0$ and that $x_1, \dots, x_n \in X$. Define $B \subset [0, 1]$ to be the set of t such that

$$\left\| \sum_{i=1}^n r_i(t) x_i \right\| > \left(\frac{16C}{\varepsilon} \right)^{1/p} \left(\sum \|x_i\|^p \right)^{1/p}.$$

Then $m(B) \leq \varepsilon/16$. However if

$$\left\| \sum r_i(t) x_i \right\| < \left(\frac{16C}{\varepsilon} \right)^{1/p} \left(\sum \|x_i\|^p \right)^{1/p}$$

then by (3.1.1) the set of $s \in K$ such that

$$\left| \sum r_i(t) Tx_i(s) \right| > \frac{1}{4} A(\varepsilon) \left(\sum \|x_i\|^p \right)^{1/p}$$

has measure at most $\varepsilon/16$.

Now let $Q \subset K \times [0, 1]$ be the set of (s, t) such that

$$\left| \sum r_i(t) Tx_i(s) \right| < \frac{1}{4} A(\varepsilon) \left(\sum \|x_i\|^p \right)^{1/p}.$$

Then $(\lambda \times m)(Q) > 1 - \varepsilon/8$. Now, by Fubini's theorem, if for $s \in K$,

$$Q_s = \{t: (s, t) \in Q\} \quad \text{and} \quad E = \left\{s: m(Q_s) > \frac{7}{8}\right\},$$

then $\lambda(E) > 1 - \varepsilon$.

Suppose $s \in E$. Then

$$m\left(t: \left| \sum r_i(t) T x_i(s) \right| \leq \frac{1}{4} \left(\sum |T x_i(s)|^2 \right)^{1/2} \right) \leq \frac{3}{4}$$

by the Paley-Zygmund inequality [9, p. 24], but $m(Q_s) > 7/8$. Hence (3.1.2) follows.

To complete the proof of the theorem we shall suppose that whenever $E \in \mathfrak{B}$ and $\lambda(E) > 1 - \varepsilon$, then there exists $x \in X$ with $\|x\| < 1$ and $\|P_E T x\|_{p,\infty} > A$. We shall deduce a contradiction and this will show that for some E , $\|P_E T\|_{p,\infty} < A$, and prove the theorem.

For each $E \in \mathfrak{B}$, let $\Gamma[E]$ be the set of pairs (x, ξ) such that $x \in X$, $\|x\| < 1$, $1 < \xi < \infty$ and $\lambda(E \cap \{|Tx| > A\xi\}) > \xi^{-p}$. Let

$$\begin{aligned} \alpha_E &= \inf\{\xi: \text{there exists } x \text{ with } (x, \xi) \in \Gamma[E]\}, \\ &= \infty \quad \text{if } \Gamma[E] = \emptyset. \end{aligned}$$

By hypothesis, $\alpha_E < \infty$ if $\lambda(E) > 1 - \varepsilon$. Note that $\alpha_E > \alpha_F$ whenever $E \subset F$.

Let $E_0 = K$ and choose $\{E_n: n = 1, 2, \dots\}$, $\{x_n: n = 1, 2, \dots\}$ and $\{\xi_n: n = 1, 2, \dots\}$ by induction as follows. Suppose E_{n-1} is given. If $\Gamma[E_{n-1}] = \emptyset$, let $x_n = 0$, $\xi_n = \infty$ and $E_n = E_{n-1}$. Otherwise choose $(x_n, \xi_n) \in \Gamma[E_{n-1}]$ with $\xi_n < 2\alpha_{E_{n-1}}$ and let

$$E_n = E_{n-1} \cap \{|Tx_n| \leq A\xi_n\}.$$

This completes the induction. Now note that $\lambda(E_n) \leq \lambda(E_{n-1}) - \xi_n^{-p}$, $n = 1, 2, \dots$, and hence $\sum \xi_n^{-p} < 1$. In particular, $\xi_n \rightarrow \infty$ and hence $\alpha_{E_n} \uparrow \infty$. Let

$$E_\infty = \bigcap_{n=1}^{\infty} E_n.$$

Then $\alpha_{E_\infty} \geq \alpha_{E_n}$ for all n and hence $\alpha_{E_\infty} = \infty$. We deduce $\lambda(E_\infty) < 1 - \varepsilon$ and so for some N , $\lambda(E_N) < 1 - \varepsilon$.

If $s \in K \setminus E_N$, then there exists $1 < n \leq N$ such that $s \in E_{n-1} \setminus E_n$ so that $\xi_n^{-1} |Tx_n(s)| > A$. Hence

$$\left(\sum_{n=1}^N \xi_n^{-2} |Tx_n(s)|^2 \right)^{1/2} > A, \quad s \in K \setminus E_N.$$

However there is a set $F \in \mathfrak{B}$ with $\lambda(F) > 1 - \varepsilon$, by (3.1.2) such that

$$\left(\sum_{n=1}^N \xi_n^{-2} |Tx_n(s)|^2 \right)^{1/2} < A \left(\sum_{n=1}^N \xi_n^{-p} \right)^{1/p} < A, \quad s \in F.$$

Thus F and $K \setminus E_N$ are disjoint but $\lambda(F) + \lambda(K \setminus E_N) > 1$. This contradiction establishes the theorem.

COROLLARY 3.2. *Under the assumptions of Theorem 3.1, there exists $\varphi \in L_0$ with $\varphi > 0$ a.e. such that $M_\varphi T \in \mathcal{L}(X, L(p, \infty))$ where $M_\varphi f = \varphi \cdot f$, $f \in L_0$.*

PROOF. For each $n \in \mathbb{N}$ pick $E_n \in \mathfrak{B}$ with $\lambda(E_n) > 1 - 1/n$ and such that $P_{E_n} T \in \mathcal{L}(X, L(p, \infty))$ with $\|P_{E_n} T\| = \beta_n$ say. Let $F_n = E_n \setminus (E_1 \cup \dots \cup E_{n-1})$

and define

$$\varphi(s) = 2^{-n}\beta_n^{-1}, \quad s \in F_n.$$

Then $M_\varphi P_{F_n} T \in \mathcal{L}(X, L(p, \infty))$ and $\|M_\varphi P_{F_n} T\| \leq 2^{-n}$. Hence

$$M_\varphi T = \sum M_\varphi P_{F_n} T \in \mathcal{L}(X, L(p, \infty)).$$

4. Local convergence of operators on L_p ($0 < p < 1$). Suppose X is a p -Banach space. We define on $\mathcal{L}(L_p, X)$ a vector topology which we call the *topology of local convergence* as follows: a base of neighborhoods of 0 consists of sets of the form

$$\mathcal{Q}(\varepsilon) = \{T: \exists B \in \mathfrak{B}, \lambda(B) > 1 - \varepsilon, \|TP_B\| < \varepsilon\},$$

for $\varepsilon > 0$. These sets form a base for a metrizable vector topology on $\mathcal{L}(L_p, X)$ weaker than the usual quasinorm topology.

PROPOSITION 4.1. *The topology of local convergence is stronger than the topology of pointwise convergence on bounded sets. The unit ball of $\mathcal{L}(L_p, X)$ is complete for local convergence.*

PROOF. Suppose $T_n \rightarrow 0$ in local convergence with $\|T_n\| \leq M < \infty$ and $f \in L_p$. There exists a sequence $\varepsilon_n \rightarrow 0$ and $B_n \in \mathfrak{B}$ with $\lambda(B_n) > 1 - \varepsilon_n$ such that $\|T_n P_{B_n}\| \leq \varepsilon_n$. Hence

$$\|T_n f\| \leq (\varepsilon_n^p \|f\|^p + \|T_n\|^p \|f - P_{B_n} f\|^p)^{1/p} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

since

$$\int_{K-B_n} |f(t)|^p d\lambda(t) \rightarrow 0, \quad \text{as } \lambda(K - B_n) \leq \varepsilon_n \rightarrow 0.$$

The second half of the proposition is an immediate consequence of the first. If $\|T_n\| \leq 1$ and T_n is a Cauchy sequence for local convergence then T_n converges pointwise to a limit T and it is easy to see that $T_n \rightarrow T$ in local convergence.

PROPOSITION 4.2. *For each $T \in \mathcal{L}(L_p, X)$ there is an essentially unique Borel function $t \mapsto \eta(t; T)$ ($t \in K$) such that*

$$\|TP_B\| = \operatorname{ess\,sup}_{t \in B} \eta(t; T)$$

whenever $B \in \mathfrak{B}$ and $\lambda(B) > 0$.

Thus $T_n \rightarrow T$ in local convergence if and only if $\eta(t; T - T_n) \rightarrow 0$ in λ -measure.

REMARK. We shall call the function $\eta(\cdot, T)$ the *local quasinorm* of T .

PROOF. For each $n \in \mathbb{N}$, let $E(n, j)$ ($1 \leq j \leq 2^n$) be a partitioning of K into 2^n disjoint Borel sets of measure 2^{-n} , such that

$$E(n, j) = E(n+1, 2j-1) \cup E(n+1, 2j), \quad 1 \leq j \leq 2^n, n = 1, 2, \dots,$$

and such that the algebra generated by the sets $\{E(n, j), 1 \leq j \leq 2^n, n = 1, 2, \dots\}$ is λ -dense in $\mathfrak{B}$.

For each $n \in \mathbb{N}$, define φ_n a simple Borel function on K by

$$\varphi_n(t) = 2^n \|T 1_{E(n, j)}\|^p, \quad t \in E(n, j).$$

Let $\mathcal{F}_n$ denote the (finite) algebra generated by $(E(n, j): 1 \leq j \leq 2^n)$. Then

$$\begin{aligned} \mathcal{E}(\varphi_{n+1} | \mathcal{F}_n)(t) &= 2^n (\|T1_{E(n+1, 2j-1)}\|^p + \|T1_{E(n+1, 2j)}\|^p) \\ &> \varphi_n(t), \quad t \in E(n, j). \end{aligned}$$

Hence φ_n is an $\mathcal{F}_n$ -submartingale and of course $0 \leq \varphi_n \leq \|T\|^p$. Thus φ_n converges λ -almost everywhere and in L^1 -norm to a limit φ . We define

$$\eta(t; T) = \varphi(t)^{1/p}, \quad t \in K.$$

If $B \in \mathcal{B}$ and $\lambda(B) > 0$, then

$$\|TP_B\| = \sup \{ \|T1_C\| / \lambda(C)^{1/p} : C \subset B, \lambda(C) > 0 \}.$$

If $C \subset B$ and $\lambda(C) > 0$, then for $\varepsilon > 0$ there exists $n \in \mathbb{N}$ and $E \in \mathcal{F}_n$ such that $\lambda(E \Delta C) < \varepsilon$. Then

$$\begin{aligned} \|T1_C\|^p &< \|T\|^p \varepsilon + \|T1_E\|^p \\ &< \|T\|^p \varepsilon + \int_E \varphi_n(t) d\lambda(t) \\ &< \|T\|^p \varepsilon + \int_E \varphi(t) d\lambda(t) \end{aligned}$$

(since φ_n is a submartingale),

$$\begin{aligned} &< 2\|T\|^p \varepsilon + \int_C \varphi(t) d\lambda(t) \\ &< 2\|T\|^p \varepsilon + \lambda(C) \operatorname{ess\,sup}_{t \in B} \varphi(t). \end{aligned}$$

Thus,

$$\|TP_B\| \leq \operatorname{ess\,sup}_{t \in B} \eta(t; T).$$

Conversely suppose, $\lambda(C) > 0$ and $\eta(t; T) > \alpha$, $t \in C$. For $\varepsilon > 0$, choose $n \in \mathbb{N}$ and $E \in \mathcal{F}_n$ as before. Thus for some $m > n$,

$$\int_C \varphi_m d\lambda > \alpha^p \lambda(C) - \varepsilon,$$

and so

$$\int_E \varphi_m d\lambda > \alpha^p \lambda(C) - \varepsilon(1 + \|T\|^p).$$

Let $E_1, \dots, E_l$ be the atoms of $\mathcal{F}_m$ contained in E .

$$\begin{aligned} \|T1_{E_j}\|^p &< \|T1_{E_j \cap C}\|^p + \|T1_{E_j \setminus C}\|^p \quad (1 \leq j \leq l) \\ &< \|TP_B\|^p \lambda(E_j \cap C) + \|T\|^p \lambda(E_j \setminus C). \end{aligned}$$

Thus,

$$\begin{aligned} \int_E \varphi_m d\lambda &< \|TP_B\|^p \lambda(E \cap C) + \|T\|^p \lambda(E \setminus C) \\ &< \|TP_B\|^p \lambda(C) + \varepsilon \|T\|^p. \end{aligned}$$

Hence

$$\|TP_B\|^p \lambda(C) \geq \alpha^p \lambda(C) - \epsilon(1 + 2\|T\|^p).$$

As $\epsilon > 0$ is arbitrary, $\|TP_B\| \geq \alpha$, and this implies

$$\|TP_B\| \geq \operatorname{ess\,sup}_{t \in B} \eta(t; T).$$

Essential uniqueness of η is obvious and the final part follows easily from the definition of local convergence.

The next result is a technical lemma whose usefulness will emerge later.

LEMMA 4.3. *Suppose $\mathcal{G}$ is a subset of $\mathcal{L}(L_p, X)$ which is closed in $\mathcal{L}(L_p, X)$ in its usual topology. Suppose $\mathcal{G}$ has the properties:*

(4.3.1) *If $T \in \mathcal{G}$ and $B \in \mathfrak{B}$ then $TP_B \in \mathcal{G}$.*

(4.3.2) *If $T \in \mathcal{L}(L_p, X)$ and for every $\epsilon > 0$ there exists $B \in \mathfrak{B}$ with $\lambda(B) > 1 - \epsilon$ and $TP_B \in \mathcal{G}$ then we have $T \in \mathcal{G}$.*

Then $\mathcal{G}$ is closed under local convergence.

PROOF. Suppose $T_n \in \mathcal{G}$ and $T_n \rightarrow T$ locally. Then by Egoroff's theorem, for any $\epsilon > 0$ there exists $B \in \mathfrak{B}$ with $\lambda(B) > 1 - \epsilon$ and

$$\operatorname{ess\,sup}_{t \in B} \eta(t; T - T_n) \rightarrow 0$$

i.e.,

$$\|(T - T_n)P_B\| \rightarrow 0.$$

Hence by (4.3.1), $TP_B \in \mathcal{G}$ and so by (4.3.2), $T \in \mathcal{G}$.

PROPOSITION 4.4. *Suppose X is a σ -complete p -Banach lattice with the Fatou property, $\|\sup_n x_n\| = \sup_n \|x_n\|$, whenever x_n is a bounded increasing sequence of positive elements (e.g. a σ -complete p -Banach function space, see §2). Then $\mathcal{L}(L_p, X)$ is a p -Banach lattice, and if $T \in \mathcal{L}(L_p, X)$ then $\||T|\| = \|T\|$ and $\eta(t; |T|) = \eta(t; T)$ a.e.*

PROOF. Clearly $\mathcal{L}(L_p, X)$ is an ordered p -Banach space. To show that it is a lattice, consider the sets $E(n, j)$ defined in 4.1. We define

$$|T|1_{E(n, j)} = \sup_{m \geq n} \sum_{k=1}^{2^{m-n}} |T1_{E(m, 2^{m-n}j - k + 1)}|,$$

where the supremum is taken in the σ -complete lattice X . The sequence on the right-hand side is increasing and bounded and we have

$$\||T|1_{E(n, j)}\|^p = \sup_{m \geq n} \left\| \sum_{k=1}^{2^{m-n}} |T1_{E(m, 2^{m-n}j - k + 1)}| \right\|^p \leq \|T\|^p \lambda(E(n, j)).$$

It is easy to see that $|T|$ may be extended to a positive linear operator in $\mathcal{L}(L_p, X)$ and that $\||T|\| = \|T\|$. Clearly, $|T| \geq \pm T$ and is the least operator with this property. Thus, $\mathcal{L}(L_p, X)$ is a p -Banach lattice.

If $B \in \mathfrak{B}$ then clearly $|T|P_B \geq \pm TP_B$ so that $|T|P_B \geq |TP_B|$ and $|T|P_{K \setminus B} \geq |TP_{K \setminus B}|$. Conversely, $|TP_B| + |TP_{K \setminus B}| \geq \pm T$ so that

$$|TP_B| + |TP_{K \setminus B}| \geq |T|P_B + |T|P_{K \setminus B}.$$

Hence, $|T|P_B = |TP_B|$ and so $\| |T|P_B \| = \| TP_B \|$. From this it follows that $\eta(t; |T|) = \eta(t; T)$ a.e.

5. Dominated operators. Let X be a quasi-Banach function space and let $T: X \rightarrow L_0$ be a linear operator. We shall say that T is dominated if there is a positive linear operator $P: X \rightarrow L_0$ such that $Pf \geq |Tf|$ a.e., $f \in X$.

Denote by $\mathfrak{M}(K)$ the space of regular Borel measures on K .

THEOREM 5.1. *Suppose X is a separable quasi-Banach function space. Then if $T \in \mathcal{L}(X, L_0)$, T is dominated if and only if there is a weak*-Borel map $t \mapsto \nu_t$, ($K \mapsto \mathfrak{M}(K)$) such that*

(5.1.1) *If $B \in \mathfrak{B}$ and $\lambda(B) = 0$ then $|\nu_t|(B) = 0$ a.e.*

(5.1.2) *If $f \in X$, then f is $|\nu_t|$ -integrable a.e.*

(5.1.3) *$Tf(t) = \int_K f(s) d\nu_t(s)$ a.e.*

PROOF. First observe that if (5.1.1), (5.1.2) and (5.1.3) hold then (observing that $t \rightarrow |\nu_t|$ is weak*-Borel, see [11]),

$$Pf(t) = \int_K f(s) d|\nu_t|(s)$$

defines a positive linear operator from X into L_0 . In fact, (5.1.2) shows that P defines a linear map. To show continuity suppose $f_n \in X$ and $\|f_n\| < 2^{-n}$; then $g = \sum |f_n| \in X$ and $Pf_n \rightarrow 0$ a.e. by applying the Dominated Convergence Theorem.

Conversely let us suppose T is dominated. As in [11, Theorem 3.1], we need only consider the case when K is totally disconnected. Suppose Σ is the countable algebra of clopen subsets of K . By modifying each on a set of measure zero we may suppose $A \mapsto T1_A(t)$ ($A \in \Sigma$), $A \mapsto P1_A(t)$ ($A \in \Sigma$) additive for all $t \in K$ and that $|T1_A(t)| \leq P1_A(t)$, $A \in \Sigma$, $t \in K$. Extend by linearity to $f \in S(\Sigma)$, the linear space of all simple continuous functions on K . Then

$$|Tf(t)| \leq |Pf(t)| \leq \|f\|_\infty P1_K(t), \quad f \in S(\Sigma).$$

Hence there exist measures $\nu_t, \mu_t \in \mathfrak{M}(K)$ such that $Tf(t) = \int_K f d\nu_t$, $f \in S(\Sigma)$, $Pf(t) = \int_K f d\mu_t$, $f \in S(\Sigma)$, and $|\nu_t| \leq \mu_t$ and $\|\mu_t\| = P1_K(t)$.

For each $f \in S(\Sigma)$, the maps $t \mapsto \int f d\nu_t$, $t \mapsto \int f d\mu_t$ are Borel and hence as $S(\Sigma)$ is dense in $C(K)$, the maps $t \mapsto \nu_t$, $t \mapsto \mu_t$ are weak*-Borel. Note that (5.1.1) and (5.1.2) must hold.

Now suppose $f \in X$ and $f \geq 0$. Then there is a sequence $f_n \in S(\Sigma)$ such that $0 \leq f_n \uparrow f$ a.e. Since X is separable, it follows that $f_n \rightarrow f$ in X (cf. [5]). By passing to a subsequence we may therefore suppose $Tf_n \rightarrow Tf$ a.e. Now $Pf_n \leq Pf$ a.e. for all n and hence by the Monotone Convergence Theorem, f is μ_t -integrable for a.e. t . Hence f is $|\nu_t|$ -integrable a.e. and so by the Dominated Convergence Theorem,

$$\begin{aligned}
Tf(t) &= \lim_{n \rightarrow \infty} Tf_n(t) \quad \text{a.e.} \\
&= \lim_{n \rightarrow \infty} \int f_n d\nu_t \quad \text{a.e.} \\
&= \int f d\nu_t \quad \text{a.e.}
\end{aligned}$$

The theorem now follows by linearity.

We shall call the map $t \mapsto \nu_t$ of 5.1 the *kernel* of T . Note that it is essentially unique.

COROLLARY 5.2. *If T is dominated there exists an operator $|T| = \sup(+T, -T)$ in $\mathcal{L}(X, L_0)$ and $|T|$ has kernel, $t \rightarrow |\nu_t|$.*

REMARK. Since $\mathcal{L}(L_1, L_1)$ is a lattice (Chacon-Krengel [1]), every $T \in \mathcal{L}(L_1, L_1)$ is dominated, and the same is true for $\mathcal{L}(L_p, L_p)$ ($0 < p < 1$) [11]. That $\mathcal{L}(L_1, L_0)$ is *not* a lattice and hence that there exist nondominated operators $T: L_1 \rightarrow L_0$ is due to Pryce [27] (Pryce only observes that $\mathcal{L}(L_1, L_0)$ is not a lattice, but the same argument holds for L_1). See also Nikisin [22].

We shall say that the kernel $t \mapsto \nu_t$ is *atomic* if $\nu_t \in \mathfrak{M}_a(K)$ a.e. where $\mathfrak{M}_a(K)$ is the space of purely atomic measures (of the form $\sum a_n \delta(t_n)$ where $\sum |a_n| < \infty$). Hence if $t \mapsto \nu_t$ is atomic it may be redefined in a Borel set of measure zero so that $\nu_t \in \mathfrak{M}_a(K)$ everywhere.

The following theorem is (essentially) proved in Theorem 3.2 of [11].

THEOREM 5.3. *If $t \mapsto \nu_t$ is an atomic kernel then there are Borel maps $a_n: K \rightarrow \mathbb{R}$, $\sigma_n: K \rightarrow K$ ($n = 1, 2, \dots$) so that*

$$|a_n(t)| \geq |a_{n+1}(t)|, \quad n = 1, 2, \dots \quad (5.3.1)$$

$$\sigma_n(t) \neq \sigma_m(t), \quad m \neq n \in \mathbb{N}. \quad (5.3.2)$$

$$\nu_t = \sum_{n=1}^{\infty} a_n(t) \delta(\sigma_n t), \quad t \in K. \quad (5.3.3)$$

DEFINITIONS. A dominated operator T will be called *elementary* if its kernel $t \mapsto \nu_t$ is atomic and the support of ν_t is (almost everywhere) at most one point; thus T is of the form

$$Tf(t) = a(t)f(\sigma t).$$

T is *locally elementary* if there exist Borel sets $(B_n: n \in \mathbb{N})$ which are disjoint and satisfy $\lambda(\bigcup B_n) = 1$ and such that TP_{B_n} is elementary for each n .

T is of *finite type* if $t \mapsto \nu_t$ is atomic and the support of ν_t is almost everywhere finite.

REMARK. T is of finite type if and only if T is the restriction of an endomorphism of L_0 (Kwapień [17], see also [11]).

THEOREM 5.4. *If T is a dominated operator of finite type and $\epsilon > 0$ then there exists $A \in \mathfrak{B}$ with $\lambda(A) \geq 1 - \epsilon$ and such that $P_A T$ is locally elementary.*

PROOF. Let $\nu_i = \sum_{n=1}^{\infty} a_n(t) \delta(\sigma_n t)$ as in Theorem 5.3. Then there exists a closed set A with $\lambda(A) > 1 - \varepsilon$ such that for some N , $a_n(t) = 0$ if $t \in A$ and $n > N$ and $\sigma_1, \dots, \sigma_N$ are continuous on A . Now each $t \in K$ has a neighborhood V_i such that the sets $A \cap \sigma_i^{-1}(V_i)$ ($1 \leq i \leq N$) are pairwise disjoint and by a compactness argument we may cover K with finitely many disjoint Borel sets $B_1, \dots, B_m$ so that the sets $A \cap \sigma_i^{-1}(B_j)$ ($1 \leq i \leq N$) are pairwise disjoint for each j . Then $P_A TP_B$ is elementary.

Now for $\mu \in \mathfrak{M}(K)$ and $x > 0$ define

$$\Delta_x(\mu) = \sum \{ \delta(t) : |\mu|(\{t\}) > x \}.$$

LEMMA 5.5. If $t \mapsto \nu_i$ is an atomic kernel then $t \mapsto \Delta_x(\nu_i)$ is weak*-Borel.

PROOF. Write $\nu_i = \sum_{n=1}^{\infty} a_n(t) \delta(\sigma_n t)$. Then

$$\Delta_x(\nu_i) = \sum_{|a_n(t)| > x} \delta(\sigma_n t)$$

is easily seen to be weak*-Borel.

For $x > 0$ define

$$\alpha_x(B) = \int_K \Delta_x(\nu_i)(B) d\lambda(t).$$

Then for each $x > 0$, α_x is a positive (possibly infinite) measure, satisfying $\lambda(B) = 0 \Rightarrow \alpha_x(B) = 0$. It follows that α_x has a (possibly infinite) Radon-Nikodym derivative. For each rational x let $w(\cdot, x)$ be a Borel derivative of α_x . Since $\alpha_x > \alpha_y$ if $x < y$, we may suppose $w(t, x) > w(t, y)$, $t \in K$, $x > y$. Now define for any $(t, x) \in K \times [0, \infty)$,

$$w(t, x) = \sup_{\substack{y > x \\ y \in Q}} w(t, y).$$

(5.5.1) w is monotone-decreasing and lower-semicontinuous in x for each fixed $t \in K$.

(5.5.2) $w(t, x)$ is a derivative of α_x for each $x > 0$.

(5.5.3) w is Borel on $K \times [0, \infty)$.

(5.5.1) is clear. To prove (5.5.2) suppose $y_n \in Q$ and $y_n \downarrow x$. Then $\alpha_{y_n}(B) \uparrow \alpha_x(B)$ for each $B \in \mathfrak{B}$ and the Monotone Convergence Theorem gives the result. For (5.5.3), observe

$$w(t, x) = \sup_m w\left(t, -\frac{1}{m}[-mx]\right),$$

where $[x]$ is the largest integer $\leq x$, and the functions

$$(t, x) \mapsto w\left(t, -\frac{1}{m}[-mx]\right)$$

are clearly Borel.

By an application of Fubini's theorem it can be seen that up to sets of product measure zero, the function w satisfying (5.5.2) and (5.5.3) is unique.

DEFINITION. The function w is called the *distribution* of the operator T (or kernel $t \mapsto v_t$).

6. Operators from L_p into $L(p, \infty)$ and L_0 where $0 < p < 1$. The space $L(1, \infty)$ is not locally convex, although Cwikel and Saghar [3] show that its dual is nontrivial. For $p < 1$, Saghar [29] and Hunt [8] show that $L(p, q)$ is r -convex whenever $r < p$ and $r < q$; in particular, $L(p, \infty)$ is r -convex for any $r < p$. The next result improves this; we note here that an inequality equivalent to Theorem 6.1 has been obtained by Pisier and Zinn (unpublished).

THEOREM 6.1. If $0 < p < 1$, $L(p, \infty)$ is p -convex.

PROOF. Let $M_p = 2^{1/p}p(1-p)^{-1}$. We shall show that if $f_1, \dots, f_n \in L(p, \infty)$,

$$\|f_1 + \dots + f_n\| \leq M_p (\|f_1\|^p + \dots + \|f_n\|^p)^{1/p}.$$

Let $h = f_1 + \dots + f_n$ and let h^* be its decreasing rearrangement. For $0 < \tau < 1$ let $A = A(\tau) = \{|h| > h^*(\tau)\}$. Then $\lambda(A) > \tau$. For $1 \leq k \leq n$, let

$$\tau_k = \frac{1}{2} \tau \frac{\|f_k\|^p}{\|f_1\|^p + \dots + \|f_n\|^p}.$$

Choose Borel sets E_k , $1 \leq k \leq n$, in K such that $\lambda(E_k) = \tau_k$ and $|f_k(t)| > f_k^*(\tau_k)$, $t \in E_k$. Let $E = E_1 \cup \dots \cup E_n$; then $\lambda(E) \leq (1/2)\tau$.

Now,

$$\begin{aligned} h^*(\tau) &\leq \inf_{t \in A} |h(t)| \leq \inf_{t \in A \setminus E} |h(t)| \\ &\leq \frac{1}{\lambda(A \setminus E)} \int_{A \setminus E} |h(t)| d\lambda(t) \leq \frac{2}{\tau} \int_{K \setminus E} |h(t)| d\lambda(t) \\ &\leq \frac{2}{\tau} \sum_{k=1}^n \int_{K \setminus E_k} |f_k(t)| d\lambda(t) \leq \frac{2}{\tau} \sum_{k=1}^n \int_{\tau_k}^{\infty} \|f_k\| x^{-1/p} dx \\ &= \frac{2p}{\tau(1-p)} \sum_{k=1}^n \|f_k\| \tau_k^{1-1/p} = \left(\frac{2}{\tau}\right)^{1/p} \frac{p}{1-p} \left(\sum_{k=1}^n \|f_k\|^p\right)^{1/p} \end{aligned}$$

and the result follows.

Now it is possible to requasinorm $L(p, \infty)$ if $0 < p < 1$ as a p -Banach function space, as in §2.

COROLLARY 6.2. $\mathcal{L}(L_p, L(p, \infty))$ and $\mathcal{L}(L_p, L_0)$ are lattices.

PROOF. By Proposition 4.4 and Corollary 3.2.

It follows that every $T \in \mathcal{L}(L_p, L_0)$ is dominated. We now show that the kernel is atomic.

THEOREM 6.3. Suppose $T \in \mathcal{L}(L_p, L_0)$; then if $p < r < 1$ the kernel v_t of T satisfies $\|v_t\|_r < \infty$ a.e. In particular, v_t is atomic.

Equivalently there exist Borel maps $a_n: K \rightarrow \mathbf{R}$, $\sigma_n: K \rightarrow K$ such that

$$Tf(t) = \sum_{n=1}^{\infty} a_n(t) f(\sigma_n t) \quad \text{a.e., } f \in L_p,$$

and

$$\sum_{n=1}^{\infty} |a_n(t)|^r < \infty \quad \text{a.e.}$$

whenever $p < r < 1$.

PROOF. By Corollary 3.2 it suffices to prove the same result for $T \in \mathcal{L}(L_p, L(p, \infty))$. As before, we may assume K totally disconnected. Let $\mathcal{Q}_n = \{A_{n,k} : 1 \leq k \leq l(n)\}$ be a partitioning of K into $l(n)$ nonempty clopen sets of diameter at most n^{-1} and suppose $\mathcal{Q}_{n+1}$ refines $\mathcal{Q}_n$ for $n \geq 1$. Pick $\tau_k^n \in A_{n,k}$ for each $1 \leq k \leq l(n)$. Let $T1_{A_{n,k}} = b_k^n$, $1 \leq k \leq l(n)$, where each b_k^n is Borel. We may further assume that

$$b_k^n = \sum_{A_{n+1,j} \subset A_{n,k}} b_j^{n+1}$$

everywhere. For $s \in K$, define

$$\nu_s^n = \sum_{k=1}^{l(n)} b_k^n(s) \delta(\tau_k^n).$$

If $p < r < 1$, $\|\nu_s^n\|_r^r = \sum_{k=1}^{l(n)} |b_k^n(s)|^r = c_n(s)$, say. Then $|b_k^n|^r \in L(pr^{-1}, \infty)$ and

$$\|c_n\|_{pr^{-1}, \infty} \leq M_{pr^{-1}} \left(\sum_{k=1}^{l(n)} \| |b_k^n|^r \|_{pr^{-1}, \infty}^{pr^{-1}} \right)^{p^{-1}}$$

by Theorem 6.1. However,

$$\| |b_k^n|^r \|_{pr^{-1}, \infty} = \|b_k^n\|_{p, \infty}^r \leq \|T\|^r \lambda(A_{n,k})^{p^{-1}}.$$

Hence,

$$\|c_n\|_{pr^{-1}, \infty} \leq M_{pr^{-1}} \|T\|^r.$$

Now $0 \leq c_n \leq c_{n+1}$ everywhere and hence, as $L(pr^{-1}, \infty)$ is σ -complete, $c = \sup c_n \in L(pr^{-1}, \infty)$ and $\|c\|_{pr^{-1}, \infty} \leq M_{pr^{-1}} \|T\|^r$. In particular, $\sup_n \|\nu_s^n\|_r < \infty$ a.e. and so almost everywhere the sequence $\{\nu_s^n : n \in \mathbb{N}\}$ is bounded. Since T is dominated, it has a kernel ν_t and clearly $\nu_t(A_{n,k}) = \lim_{m \rightarrow \infty} \nu_t^m(A_{n,k})$ a.e., for each k, n . Hence $\nu_t^m \rightarrow \nu_t$ weak* almost everywhere. In particular,

$$\|\nu_t\|_r \leq \liminf_{m \rightarrow \infty} \|\nu_t^m\|_r < \infty \quad \text{a.e.}$$

The result now follows from Theorem 5.1.

EXAMPLE. Suppose $(\xi_t : 0 \leq t \leq 1)$ is a symmetric p -stable process in $L_0(K)$, i.e. if $0 \leq t_1 < t_2 < \dots < t_n \leq 1$ then $\xi_{t_j} - \xi_{t_{j-1}}$ ($2 \leq j \leq n$) are mutually independent and

$$\mathbb{E}(e^{i\tau(\xi_t - \xi_s)}) = \exp(-|t - s| |\tau|^p).$$

Then, $Tf = \int f d\xi_t$ defines an embedding of L_p into L_0 . The preceding theorem guarantees that the sample paths $\xi_t(s)$ ($0 \leq t \leq 1$) almost everywhere are jump functions with countably many jumps $a_n(s)$ such that $\sum |a_n(s)|^r < \infty$ for any $r > p$.

Theorem 6.3 does not give necessary and sufficient conditions for $T \in \mathcal{L}(L_p, L(p, \infty))$. We now proceed to this problem.

THEOREM 6.4. *In order that the weak*-Borel map $t \mapsto \nu_t$ be the kernel of a linear operator $T \in \mathcal{L}(L_p, L(p, \infty))$ it is necessary and sufficient that for some C ,*

$$x\alpha_x(B)^{1/p} \leq C\lambda(B)^{1/p}, \quad B \in \mathfrak{B}. \quad (6.4.1)$$

Then,

$$\|T\| \leq M_p C. \quad (6.4.2)$$

REMARK. Of course,

$$\alpha_x(B) = \int_K \Delta_x(\nu_t)[B] d\lambda(t), \quad B \in \mathfrak{B}.$$

PROOF. First suppose $T \in \mathcal{L}(L_p, L(p, \infty))$. Then its kernel ν_t is atomic. Suppose as in Theorem 5.3, $\nu_t = \sum_{n=1}^{\infty} a_n(t)\delta(\sigma_n t)$. Then $|T|$ has kernel $|\nu_t| = \sum_{n=1}^{\infty} |a_n(t)|\delta(\sigma_n t)$ and $\| |T| \| \leq M_p \|T\|$. Let

$$\mu_t^n = \sum_{k=1}^n |a_k(t)|\delta(\sigma_k t).$$

Then μ_t^n is the kernel of a positive linear operator $S_n \in \mathcal{L}(L_p, L(p, \infty))$ of finite type. Pick A_n with $\lambda(A_n) \geq 1 - 1/n$ so that $P_{A_n}S_n$ is locally elementary. Suppose $P_{A_n}S_nP_{B_j}$ is elementary for $1 \leq j < \infty$ where $\bigcup_j B_j = K$.

Suppose $B \subset B_j$ for some j ; let $P_{A_n}S_n1_B = h$. Then

$$P_{A_n}S_n1_B = \sum_{k=1}^n a_k \cdot 1_{\sigma_k^{-1}(B)} \cdot 1_{A_n}$$

and as the sets $A_n \cap \sigma_k^{-1}(B)$ are mutually disjoint,

$$\lambda(|h| > x) = \sum_{k=1}^n \lambda(\sigma_k^{-1}B \cap A_n \cap (|a_k| > x)). \quad (6.4.3)$$

Hence

$$x^p \sum_{k=1}^n \lambda(\sigma_k^{-1}B \cap A_n \cap (|a_k| > x)) \leq M_p^p \|T\|^p \lambda(B).$$

Letting $n \rightarrow \infty$ we have

$$x^p \sum_{k=1}^{\infty} \lambda(\sigma_k^{-1}B \cap (|a_k| > x)) \leq M_p^p \|T\|^p \lambda(B),$$

i.e.,

$$x(\alpha_x(B))^{1/p} \leq M_p \|T\| \lambda(B)^{1/p}.$$

For the converse, define μ_t^n, A_n as before. We show that the formally defined operator $P_{A_n}S_n \in \mathcal{L}(L_p, L(p, \infty))$ and $\|P_{A_n}S_n\| \leq CM_p$. This follows easily from (6.4.3) and the fact that $L(p, \infty)$ is p -convex. Then $P_{A_n}S_n f \uparrow P f$ for $f \geq 0$ and P is a positive operator with $\|P\| \leq CM_p$. P has kernel $|\nu_t|$ and it then follows that ν_t is the kernel of an operator T with $|T| = P$. Hence, $\|T\| \leq CM_p$.

COROLLARY 6.5. *In order that v_t be the kernel of some $T \in \mathcal{L}(L_p, L(p, \infty))$ it is necessary and sufficient that*

$$\operatorname{ess\,sup}_{t \in K} W^{p,\infty}(t) = C < \infty, \quad (6.5.1)$$

where

$$W^{p,\infty}(t) = \sup_{0 < x < \infty} xw(t, x)^{1/p}, \quad (6.5.2)$$

where w is the distribution of v_t . Then

$$\|T\| \leq CM_p.$$

PROOF. If $\alpha_x(B) \leq C^p x^{-p} \lambda(B)$, $B \in \mathfrak{B}$, then $w(t, x) \leq C^p x^{-p}$, λ -a.e. for each $x > 0$. By Fubini's theorem, for λ -a.e. t , $x^p w(t, x) \leq C^p$, x -a.e., but as w is monotone in x this means $x^p w(t, x) \leq C^p$. Hence $W^{p,\infty}(t) \leq C$, λ -a.e.

The converse is easy.

COROLLARY 6.6. *Suppose $T_n \in \mathcal{L}(L_p, L(p, \infty))$ have distributions w_n and*

$$W_n^{p,\infty}(t) = \sup_{0 < x < \infty} xw_n(t, x)^{1/p}.$$

Then if $W_n^{p,\infty}(t) \rightarrow 0$ in λ -measure, $T_n \rightarrow 0$ in the topology of local convergence.

PROOF. If $B \in \mathfrak{B}$, then $T_n P_B$ has distribution

$$\begin{aligned} w_{n,B}(t, x) &= w_n(t, x), & t \in B, \\ &= 0, & t \notin B, \end{aligned}$$

and hence,

$$\|T_n P_B\| \leq M_p \operatorname{ess\,sup}_{t \in B} W_n^{p,\infty}(t).$$

The result now follows from Egoroff's theorem.

7. Operators from L_p into L_0 ; the main result.

PROPOSITION 7.1. *Suppose $0 < p < 1$ and $p < q < r \leq 2$. Suppose $T \in \mathcal{L}(L_p, L(p, \infty))$ is nonzero and has distribution w satisfying*

$$\lim_{x \rightarrow 0} xw(t, x)^{1/p} = \lim_{x \rightarrow \infty} xw(t, x)^{1/p} = 0, \quad \lambda\text{-a.e.} \quad (7.1.1)$$

Then there is a strongly embedded subspace V of L_q such that $V \cong L_r$ and $T|V$ is an isomorphism.

[V is strongly embedded in L_q if the L_q - and L_0 -topologies agree on V .]

PROOF. Consider the following property of operators $T \in \mathcal{L}(L_p, L(p, \infty))$.

(7.1.2) *There exists $c > 0$ and $B \in \mathfrak{B}$ with $\lambda(B) > 0$ such that whenever $C \in \mathfrak{B}$ with $C \subset B$ and $\lambda(C) > 0$ then there is a strongly embedded subspace $V[C]$ of $L_q(C)$ such that $V_C \cong L_r$ and $\|Tf\|_{p,\infty} \geq c\|f\|_p$, $f \in V[C]$.*

Let $\mathcal{G}$ be the subset of $\mathcal{L}(L_p, L(p, \infty))$ of all T for which (7.1.2) fails to hold. $\mathcal{G}$ is certainly closed and satisfies the condition of Lemma 4.3. Hence $\mathcal{G}$ is closed also for local convergence.

Suppose $\mathcal{G}$ contains an elementary operator $S \neq 0$, say $Sf(t) = a(t)f(\sigma t)$. Then there is a subset A of K with $\lambda(A) > 0$ such that $0 < \varepsilon \leq |a(t)| \leq 2\varepsilon$, $t \in A$, and then the measure $\mu(B) = \lambda(\sigma^{-1}B \cap A)$, $B \in \mathfrak{B}$, is λ -continuous, with λ -derivative φ say. Choose $E \in \mathfrak{B}$ with $\lambda(E) > 0$ so that $0 < \delta \leq \varphi(t) \leq 2\delta$, $t \in E$.

Now let Λ be the standard embedding of L_r into L_q by using an r -stable process (cf. [16] and the Example after 6.3; also cf. [8] for an order-preserving embedding). Observe that Λ embeds L_r strongly into L_q and that for some constant γ depending only on p , $\|\Lambda f\|_{p,\infty} = \gamma\|\Lambda f\|_p$, $f \in L_r$. Suppose $C \in \mathfrak{B}$, $C \subset E$ and $\lambda(C) > 0$. Let $\theta_C: C \rightarrow K$ be any Borel map such that $\lambda(\theta_C^{-1}(B)) = \lambda(C)\lambda(B)$, $B \in \mathfrak{B}$. Define $R_C: L_0(K) \rightarrow L_0(K)$ by

$$\begin{aligned} R_C f &= f(\theta_C t), & t \in C, \\ &= 0, & t \notin C. \end{aligned}$$

Let $V = V[C] = R_C \Lambda(L_r)$. Then $V[C]$ is isomorphic to L_r and strongly embedded in $L_q(C)$. If $f \in V[C]$, $\|f\|_{p,\infty} = \gamma\|f\|_p$. For $x > 0$, $f \in V[C]$,

$$\begin{aligned} \lambda(|Sf| > x) &\geq \lambda(A \cap (|f \circ \sigma| > x\varepsilon^{-1})) \\ &= \mu(|f| > x\varepsilon^{-1}) \geq \delta\lambda(|f| > x\varepsilon^{-1}). \end{aligned}$$

Hence,

$$\begin{aligned} x\lambda(|Sf| > x)^{1/p} &\geq x\delta^{1/p}\lambda(|f| > x\varepsilon^{-1})^{1/p} \\ &= \varepsilon\delta^{1/p}(x\varepsilon^{-1})\lambda(|f| > x\varepsilon^{-1})^{1/p}. \end{aligned}$$

Hence,

$$\|Sf\|_{p,\infty} \geq \varepsilon\delta^{1/p}\|f\|_{p,\infty} = \gamma\varepsilon\delta^{1/p}\|f\|_p.$$

This shows (7.1.2) does hold with $c = \gamma\varepsilon\delta^{1/p}$ and $B = E$. We conclude $S = 0$.

Now suppose $T \in \mathcal{G}$ has kernel ν_t , where $\nu_t = \sum_{n=1}^{\infty} a_n(t)\delta(\sigma_n t)$ as in Theorem 5.3, and the distribution w of T satisfies (7.1.1).

If $T \neq 0$ then $S \neq 0$ where S has kernel $a_1(t)\delta(\sigma_1 t)$; of course $S \in \mathcal{L}(L_p, L(p, \infty))$ and $|S| \leq |T|$.

For $n \in \mathbb{N}$, let $\mathfrak{B}_n = \{B_{n,1}, \dots, B_{n,l(n)}\}$ be a partition of K into $l(n)$ disjoint Borel sets of diameter at most n^{-1} . Assume $\mathfrak{B}_{n+1}$ refines $\mathfrak{B}_n$. Let $C_{n,i} = \sigma_1^{-1}B_{n,i}$ and

$$T_n = \sum_{i=1}^{l(n)} P_{C_{n,i}} T P_{B_{n,i}}.$$

Then $T_n \in \mathcal{G}$. T_n has kernel ν_t^n where $\nu_t^n(B) = \nu_t(B \cap B_{n,i})$, $t \in C_{n,i}$. For fixed t , if $t \in C_{n,i(n)}$,

$$\begin{aligned} \|\nu_t^n - a_1(t)\delta(\sigma_1 t)\| &= \|\nu_t^n - a_1(t)\delta(\sigma_1 t)\|(K) \\ &= \|\nu_t - a_1(t)\delta(\sigma_1 t)\|(B_{n,i(n)}) \end{aligned}$$

and so

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|\nu_t^n - a_1(t)\delta(\sigma_1 t)\| &\leq \|\nu_t - a_1(t)\delta(\sigma_1 t)\| \limsup_{n \rightarrow \infty} \lambda(B_{n,i(n)}) \\ &= \|\nu_t - a_1(t)\delta(\sigma_1 t)\| \{\sigma_1 t\} = 0. \end{aligned}$$

Thus, for $x > 0$,

$$\Delta_x(\nu_t^n - a_1(t)\delta(\sigma_1 t)) \rightarrow 0$$

and $\Delta_x(\nu_t^n - a_1(t)\delta(\sigma_1 t))$ is monotone decreasing.

For $x > 0$, let

$$\alpha_x^n(B) = \int_K \Delta_x(\nu_t^n - a_1(t)\delta(\sigma_1 t))(B) d\lambda(t).$$

Since

$$\Delta_x(\nu_t^n - a_1(t)\delta(\sigma_1 t)) \leq \Delta_x(\nu_t),$$

we may apply the Dominated Convergence Theorem to deduce $\alpha_x^n(B) \downarrow 0$ for $B \in \mathfrak{B}$.

Now let w_n be the distribution of $T_n - S$. We can assume w_n monotone decreasing and since $\int_B w_n(t, x) d\lambda(t) = \alpha_x^n(B)$, $B \in \mathfrak{B}$, we have $w_n(t, x) \downarrow 0$ a.e. $(t, x) \in K \times [0, \infty)$ (apply Fubini's theorem). Hence, for λ -a.e. t , $w_n(t, x) \downarrow 0$, $0 < x < \infty$. Now fix t and let

$$\bar{w}_n(t, x) = \limsup_{y \rightarrow x} w_n(t, y).$$

Then

$$\bar{w}_n(t, x) \leq w_n\left(t, \frac{1}{2}x\right), \quad 0 < x < \infty,$$

and so

$$\bar{w}_n(t, x) \downarrow 0$$

and each $\bar{w}_n$ is upper-semicontinuous. Hence so is $x\bar{w}_n(t, x)^{1/p}$. By Dini's theorem,

$$x\bar{w}_n(t, x)^{1/p} \downarrow 0$$

uniformly on compact subsets of $(0, \infty)$. By (7.1.1), we can conclude convergence is uniform on $(0, \infty)$, i.e. $W_n^{p,\infty}(t) \rightarrow 0$, λ -a.e. Hence $T_n \rightarrow S$ in local convergence and so $S \in \mathcal{G}$. This contradiction proves the proposition.

THEOREM 7.2. Suppose $0 < p < 1$ and $T \in \mathcal{L}(L_p, L_0)$ with $T \neq 0$. Then if $p < r < 2$, there is a subspace V of L_p isomorphic to L_r and such that T is an isomorphism on V .

PROOF. We may suppose $r < 1$. By Nikisin's theorem there is a subset B of K such that $S = P_B T \neq 0$, $S \in \mathcal{L}(L_p, L(p, \infty))$ and $S \in \mathcal{L}(L_r, L(r, \infty))$. Now choose $p < q_1 < q_2 < r$.

Let w be the distribution of S . Then

$$\sup_{0 < x < \infty} x^p w(t, x) < C_1 \quad \text{a.e.,}$$

and

$$\sup_{0 < x < \infty} x^r w(t, x) < C_2 \quad \text{a.e.}$$

Hence if $q = q_1$ or q_2 ,

$$\begin{aligned} x^q w(t, x) &\leq \min(C_1 x^{q-p}, C_2 x^{q-r}) \\ &\leq C_1 x^{q-p}, \quad 0 \leq x \leq 1, \\ &\leq C_2 x^{q-r}, \quad 1 \leq x < \infty. \end{aligned}$$

Hence,

$$x^q w(t, x) \leq \max(C_1, C_2), \quad 0 \leq x < \infty,$$

and

$$\lim_{x \rightarrow \infty} x w(t, x)^{1/q} = \lim_{x \rightarrow \infty} x w(t, x)^{1/q} = 0.$$

Thus $S \in \mathcal{L}(L_q, L(q, \infty))$.

By Proposition 7.1, there is a strongly embedded subspace V of L_{q_2} isomorphic to L_r such that T maps V isomorphically from L_{q_1} into $L(q_1, \infty)$. Since $V \subset L_{q_2}$, $T(V) \subset L(q_2, \infty)$. The $L(q_2, \infty)$ -topology on $T(V)$ is stronger than the $L(q_1, \infty)$ -topology, but the L_{q_2} - and L_{q_1} -topologies agree on V . Hence the $L(q_1, \infty)$ - and $L(q_2, \infty)$ -topologies and all intermediate L_s -topologies agree on $T(V)$, i.e., $T(V)$ is strongly embedded in $L(q_1, \infty)$. Hence T maps V isomorphically into L_0 (of course the L_{q_2} - and L_p -topologies agree on V).

8. Embedding l_p and L_p in p -Banach function spaces. Let X be a p -Banach function space. Then we can apply Nikisin's theorem (Corollary 3.2) to deduce that there is a function $\varphi \in L_0$, with $\varphi > 0$ such that $M_\varphi(X) \subset L(p, \infty)$. It follows that we can, without loss of generality, suppose that every p -Banach function space considered is contained in $L(p, \infty)$ and $\|f\|_X \geq \|f\|_{p,\infty}$, $f \in X$ [simply replace X by $M_\varphi(X)$].

THEOREM 8.1. *Suppose X is a p -Banach function space satisfying:*

(8.1.1) *There exists $c > 0$ and $r > 0$ such that whenever $f_1, \dots, f_n \in X$ and $|f_i| \wedge |f_j| = 0$, $i \neq j$, then*

$$\|f_1 + \dots + f_n\| \geq c \left(\sum \|f_i\|^r \right)^{1/r}.$$

Then if X contains a subspace isomorphic to l_p , there is a sequence of positive elements with disjoint support equivalent to the usual basis of l_p .

REMARK. Equivalently l_p embeds in X as a sublattice.

PROOF. Suppose (8.1.1) holds and that Y is a subspace of X isomorphic to l_p . We consider two possibilities: (a) Y fails to be strongly embedded in X , and (b) Y is strongly embedded in X .

(a) In this case Y contains a sequence $g_n \rightarrow 0$ in L_0 but such that $\|g_n\| \geq \varepsilon > 0$ and $\{g_n\}$ is equivalent to the usual basis of l_p . Here $\{g_n\}$ can be obtained as a block basis of the original basis of Y , by a standard gliding hump argument.

Observe that (8.1.1) implies that whenever $0 < f_n \uparrow f$ a.e. in X then $\|f - f_n\| \rightarrow 0$. Hence if $f \geq 0$ and $f \in X$ and $A_n \in \mathfrak{B}$ with $\lambda(A_n) \rightarrow 0$ then $\|f \cdot 1_{A_n}\| \rightarrow 0$.

Now choose an increasing sequence of integers $\{m(n)\}$, a decreasing sequence $\{\varepsilon_n\}$, and a sequence $A_n \in \mathfrak{B}$ such that

$$\lambda(A_n) < \varepsilon_n, \quad n = 1, 2, \dots, \quad (8.1.2)$$

$$\|g_{m(n)} 1_{K-A_n}\| < \varepsilon_n, \quad n = 1, 2, \dots, \quad (8.1.3)$$

$$0 < 2\varepsilon_{n+1} < \varepsilon_n, \quad n = 1, 2, \dots, \quad (8.1.4)$$

$$\text{if } \lambda(A) < 2\varepsilon_{n+1}, \|g_{m(n)} \cdot 1_A\| < \varepsilon_n, \quad n = 1, 2, \dots. \quad (8.1.5)$$

Pick $\varepsilon_1 = 1$. Now suppose $\varepsilon_1, \dots, \varepsilon_n, A_1, \dots, A_{n-1}, m(1), \dots, m(n-1)$ have been chosen where $n \geq 1$.

Let $h = \sum 2^{-n} |g_n|$; $h \in X$ and $h \geq 0$. Now by Egoroff's theorem, since $g_n \rightarrow 0$ in measure, there exists $m(n) > m(n-1)$ and $A_n \in \mathcal{B}$ with $\lambda(A_n) < \varepsilon_n$ and

$$|g_{m(n)}| 1_{K-A_n} < \varepsilon_n \|h\|^{-1} h.$$

Then

$$\|g_{m(n)} 1_{K-A_n}\| < \varepsilon_n.$$

Now choose ε_{n+1} so that $\varepsilon_{n+1} > 0$ and (8.1.4) and (8.1.5) hold. This completes the induction.

Let $B_n = A_n \setminus \bigcup_{k>n} A_k$ and $f_n = g_{m(n)} \cdot 1_{B_n}$. Then

$$\|g_{m(n)} L - f_n\|^p < \|g_{m(n)} \cdot 1_{K-A_n}\|^p + \|g_{m(n)} \cdot 1_{\bigcup_{k>n} A_k}\|^p < 2\varepsilon_n^p$$

since $\lambda(\bigcup_{k>n} A_k) < 2\varepsilon_{n+1}$. Hence,

$$\|g_{m(n)} - f_n\| \rightarrow 0$$

and so, passing to a subsequence, we may assume $\{f_n\}$ equivalent to the usual l_p -basis.

(b) We shall show that condition (b) leads to a contradiction. Let $\{f_n\}$ be equivalent to the usual l_p -basis in Y and hence also in $L(p, \infty)$. Let $V = \{f_n: n \in \mathbb{N}\}$ and suppose

$$M = \sup_n \|f_n\|_X$$

and

$$\|g_1 + \dots + g_n\|_{p,\infty} \geq an^{1/p}$$

whenever $g_1, \dots, g_n$ are distinct elements of V , where $a > 0$.

Let u be chosen so that $u > 1$ and

$$u^{1/p-1} > \frac{2^{(1/p)+2} Mp}{a(1-p)}$$

and let

$$\alpha = \frac{a}{(u-1)} 2^{-(2/p)-2}.$$

For each k , let A_k be the set of integers θ_n^k such that $\theta_1^k < \theta_2^k < \dots$ and

$$2^{-\theta_k^k/p} f_k^*(2^{-\theta_k^k}) \geq 2^{-2/p} \alpha.$$

We claim $|A_k| < \infty$ and $|A_k| < M^r 2^{3r/p} \alpha^{-1} C^{-r}$. Indeed, suppose $\theta_1^k < \theta_2^k < \dots < \theta_m^k \in A_k$; then

$$f_k = \gamma_1 + \dots + \gamma_m$$

where the γ_i 's have disjoint support and

$$\|\gamma_i\|_{p,\infty} > 2^{-3/p}\alpha.$$

Hence,

$$\|f_k\|_X > cm^{1/r}2^{-3/p}\alpha$$

and so

$$m < M^r 2^{3r/p} \alpha^{-1} c^{-r}.$$

Now we may pass to a subsequence of f_k (also called f_k) such that $|A_k| = L$ is constant and for some $l < L$, $\theta_i^k = \theta_i$, $i < l$, $k = 1, 2, \dots$, and $\theta_i^{k+1} > \theta_i^k$, $i > l$, $k = 1, 2, \dots$. Hence if $\rho > \theta_l$, ρ belongs to at most one set A_k . Now suppose $n > 2^{\theta_l}$, and let

$$h_n = f_1 + \dots + f_n.$$

For some $\tau = \tau_n$, $0 < \tau < 1$,

$$\tau^{1/p} h_n^*(\tau) > \frac{1}{2} an^{1/p}.$$

Let $A = \{|h_n| > h_n^*(\tau)\}$ and choose $E_1, \dots, E_n \in \mathfrak{B}$ so that $\lambda(E_i) = \tau/2n$ ($1 < i < n$) and $|f_i(t)| > f_i^*(\tau/2n)$, $t \in E_i$, $1 < i < n$. Let $E = E_1 \cup \dots \cup E_n$ so that $\lambda(E) < \tau/2$ and $\lambda(A) > \tau$.

$$\begin{aligned} \frac{1}{2} an^{1/p} &< \tau^{1/p} h_n^*(\tau) \\ &< \frac{\tau^{1/p}}{\lambda(A \setminus E)} \int_{A \setminus E} |h_n(t)| d\lambda(t) \quad (\text{as in Theorem 6.1}) \\ &< 2\tau^{1/p-1} \sum_{k=1}^n \int_{K \setminus E_k} |f_k(t)| d\lambda(t) \\ &< 2\tau^{1/p-1} \sum_{k=1}^n \int_{\tau/2n}^1 f_k^*(x) dx \\ &< 2\tau^{1/p-1} \sum_{k=1}^n \left\{ \int_{\tau/2n}^{u\tau/2n} f_k^*(x) dx + \int_{u\tau/2n}^\infty \|f_k\| x^{-1/p} dx \right\} \\ &< \frac{(u-1)\tau^{1/p}}{n} \sum_{k=1}^n f_k^*\left(\frac{\tau}{2n}\right) + 2^{1/p} \frac{n^{1/p}}{u^{1/p-1}} \cdot \frac{Mp}{1-p} \\ &< \frac{(u-1)\tau^{1/p}}{n} \sum_{k=1}^n f_k^*\left(\frac{\tau}{2n}\right) + \frac{1}{4} an^{1/p}. \end{aligned}$$

Hence,

$$\frac{1}{4} an^{1/p} < \frac{(u-1)\tau^{1/p}}{n} \sum_{k=1}^n f_k^*\left(\frac{\tau}{2n}\right).$$

Pick ρ_n to be an integer such that $2^{-\rho_n} < \tau/2n < 2^{1-\rho_n}$. Then $\tau^{1/p} < 2^{2/p-\rho_n/p} n^{1/p}$ and so

$$\frac{1}{4} an^{1/p} < 2^{2/p-\rho_n/p} (u-1) n^{1/p-1} \sum_{k=1}^n f_k^*(2^{-\rho_n}).$$

Hence,

$$\alpha n \leq 2^{-\rho_n/p} \sum_{k=1}^n f_k^*(2^{-\rho_n}).$$

Now $\rho_n \geq \log_2(2n/\tau) \geq \log_2(2n) > \theta_l$. Hence ρ_n belongs to at most one set A_k .

Thus, $2^{-\rho_n/p} \sum_{k=1}^n f_k^*(2^{-\rho_n}) \leq (n-1)\alpha 2^{-2/p} + M$. Thus, $\alpha \leq 2^{-2/p}\alpha(1-1/n) + M/n$ for all $n > 2^{\theta_l}$. This leads to a contradiction, and the theorem is proved.

We now turn to the problem of embedding L_p in X . We first prove some preparatory lemmas; these will also be useful in the next section.

LEMMA 8.2. *Suppose X is a p -Banach function space where $0 < p < 1$. Then:*

- (i) *Any $T \in \mathcal{L}(L_p, X)$ has an atomic kernel.*
- (ii) *If $S, T \in \mathcal{L}(L_p, X)$ have kernels ν_i and μ_i respectively, then the kernel of $S \vee T$ is $\nu_i \vee \mu_i$ almost everywhere.*

PROOF. (i) Since $X \subset L(p, \infty)$, this follows from Theorem 6.3.

(ii) This follows from Corollary 5.2 since $S \vee T = (1/2)(S + T + |S - T|)$. Of course, $|S - T|$ in $\mathcal{L}(L_p, L_0)$ is identical with $|S - T|$ in $\mathcal{L}(L_p, X)$.

LEMMA 8.3. *Suppose X is a p -Banach function space with $0 < p < 1$. Suppose $T_n, T \in \mathcal{L}(L_p, X)$ and $0 < T_n < T_{n+1} < T$ for $n \geq 1$. Suppose T has kernel μ_i and T_n has kernel μ_i^n . Then if $\mu_i^n \rightarrow \mu_i$ weak* a.e., $\eta(t; T) = \lim_{n \rightarrow \infty} \eta(t; T_n)$ a.e.*

PROOF. Clearly $\eta(t; T_n)$ is monotone increasing almost everywhere. If $\lambda(B) > 0$ and $\varepsilon > 0$, $\eta(t; T_n) + \varepsilon \leq \eta(t; T)$, $t \in B$, $n \in \mathbb{N}$ then, $\|T_n P_B\| + \varepsilon \leq \|TP_B\|$, $n \in \mathbb{N}$.

Select $B_1 \subset B$ with $\lambda(B_1) > 0$ such that

$$\|T_1 1_{B_1}\| > \left(\|TP_B\| - \frac{1}{2}\varepsilon \right) \lambda(B_1)^{1/p}.$$

Then since $\mu_i^n \leq \mu_i^{n+1} \leq \mu_i$ we have $\|\mu_i^n - \mu_i\| \rightarrow 0$ a.e. and hence $T_n 1_{B_1}(t) \rightarrow T 1_{B_1}(t)$, a.e. Now by the Fatou property of the quasinorm, there exists n so that

$$\|T_n 1_{B_1}\| > (\|TP_B\| - \varepsilon) \lambda(B_1)^{1/p}$$

and hence

$$\|T_n P_B\| > \|TP_B\| - \varepsilon.$$

This contradiction proves the result.

LEMMA 8.4. *Suppose $T \in \mathcal{L}(L_p, L_0)$. Then there is a sequence of operators $(S_n, n \in \mathbb{N})$ such that*

$$|S_i| \wedge |S_j| = 0, \quad i \neq j. \quad (8.4.1)$$

$$S_1 + \cdots + S_n \text{ is locally elementary for each } n. \quad (8.4.2)$$

If ν_i^n is the kernel of S_n and μ_i is the kernel of T , then $\sum_{n=1}^{\infty} \nu_i^n = \nu_i$ weak a.e., and $\sum_{n=1}^{\infty} |\nu_i^n| = |\nu_i|$ weak* a.e.*

PROOF. Let

$$\nu_i = \sum_{n=1}^{\infty} a_n(t) \delta(\sigma_n t)$$

where as in Theorem 5.3, each $a_n: K \rightarrow \mathbf{R}$ is Borel and each $\sigma_n: K \rightarrow K$ is Borel, and $\sigma_i(t) \neq \sigma_j(t)$ whenever $i \neq j$. Now let

$$\mu_i^n = \sum_{k=1}^n a_k(t) \delta(\sigma_k t).$$

Then μ_i^n is the kernel of an operator $T_n \in \mathcal{L}(L_p, L_0)$ with $|T_n| < |T|$. Further, T_n is of finite type. By Theorem 5.4 we may select $A_n \in \mathcal{B}$ with $\lambda(A_n) > 1 - 2^{-(n+1)}$ such that $P_{A_n} T_n$ is locally elementary. Let $B_n = \bigcap (A_k: k > n)$; then $\lambda(B_n) > 1 - 2^{-n}$ and $P_{B_n} T_n$ is locally elementary. Finally, let $S_1 = P_{B_1} T_1$ and $S_n = P_{B_n} T_n - P_{B_{n-1}} T_{n-1}$ ($n > 2$). Then S_n has kernel ν_i^n given by

$$\begin{aligned} \nu_i^n &= 0, & t \notin B_n, \\ &= \mu_i^n, & t \in B_n \setminus B_{n-1}, \\ &= \mu_i^n - \mu_i^{n-1}, & t \in B_{n-1}. \end{aligned}$$

Clearly, $|\nu_i^n| \wedge |\nu_i^m| = 0$ a.e., $m \neq n$, and (8.4.1), (8.4.2), and (8.4.3) follow easily.

LEMMA 8.5. Suppose X is a p -Banach function space satisfying (8.1.1) and that $T_1, \dots, T_n \in \mathcal{L}(L_p, X)$ satisfy $|T_i| \wedge |T_j| = 0$, $i \neq j$. Then,

$$\eta(t; T_1 + \dots + T_n) > c \left\{ \sum_{i=1}^n \eta(t; T_i) \right\}^{1/r} \quad \text{a.e.} \quad (8.5.1)$$

PROOF. Suppose first $S = T_1 + \dots + T_n$ is elementary. Then by considering kernels we see that if $|T_i| \wedge |T_j| = 0$ ($i \neq j$) then for any $f \in L_p$,

$$|T_i f| \wedge |T_j f| = 0.$$

Hence, for any Borel set B ,

$$\|S 1_B\| > c \left(\sum \|T_i 1_B\| \right)^{1/r}$$

and (8.5.1) follows from the construction of the local quasinorm η as in Proposition 4.2.

It now quickly follows that (8.5.1) holds for locally elementary S . For general S , let (S_n) be chosen as in Lemma 8.4. Then if

$$V_m = |S_1| + \dots + |S_m|,$$

then

$$V_m = V_m \wedge |T_1| + \dots + V_m \wedge |T_n|,$$

and since V_m is locally elementary,

$$\eta(t; V_m) > c \left(\sum_{i=1}^n \eta(t; V_m \wedge |T_i|) \right)^{1/r} \quad \text{a.e.}$$

Now by Lemma 8.3,

$$\lim_{m \rightarrow \infty} \eta(t; V_m) = \eta(t; |S|) \quad \text{a.e.}$$

and a similar argument shows

$$\lim_{m \rightarrow \infty} \eta(t; V_m \wedge |T_i|) = \eta(t; |T_i|) \quad \text{a.e.}$$

The lemma now follows.

LEMMA 8.6. *Suppose X is a p -Banach function space ($0 < p < 1$) satisfying (8.1.1). Then the locally elementary operators are dense in $\mathcal{L}(L_p, X)$ in the topology of local convergence.*

PROOF. Suppose $S \in \mathcal{L}(L_p, X)$. We appeal to Lemma 8.4 to define (S_n) satisfying (8.4.1), (8.4.2), and (8.4.3). Suppose $m(n)$ is an increasing sequence of integers with $m(0) = 0$; then let

$$T_n = \sum_{m(n-1)+1}^{m(n)} S_i.$$

By Lemma 8.5,

$$\eta(t; T_1 + \cdots + T_n) > c \left(\sum_{i=1}^n \eta(t; T_i)^r \right)^{1/r} \quad \text{a.e.}$$

Hence,

$$c \left(\sum \eta(t; T_i)^r \right)^{1/r} < \eta(t; |S|) \quad \text{a.e.,}$$

and hence,

$$\eta(t; T_n) \rightarrow 0 \quad \text{a.e.}$$

Thus $\sum_{i=1}^n S_i$ is a Cauchy sequence in the topology of local convergence, and as $|\sum_{i=1}^n S_i| < |T|$ by Proposition 4.1, $\sum_{i=1}^\infty S_i$ converges in this topology. Clearly, $\sum_{i=1}^\infty S_i = S$. To see this observe that given $\varepsilon > 0$ there exists $B_k \in \mathfrak{B}$ with $\lambda(B) > 1 - 1/k$ and $\|(\sum_{i=1}^\infty S_i - \sum_{i=1}^n S_i)P_{B_k}\| \rightarrow 0$ as $n \rightarrow \infty$. Thus,

$$\left| \sum_{i=1}^\infty S_i - \sum_{i=1}^n S_i \right| 1_{B_k} \rightarrow 0.$$

Hence, if μ_t is the kernel of $\sum_{i=1}^\infty S_i$, $|\mu_t - \sum_{i=1}^n \nu_i| (B_k) \rightarrow 0$ as $n \rightarrow \infty$, i.e., $|\mu_t - \nu_t| (B_k) = 0$ a.e. Thus, $|\mu_t - \nu_t| (\cup_{k=1}^\infty B_k) = 0$ a.e. As $\lambda(K \setminus \cup B_k) = 0$, $|\mu_t - \nu_t| (K \setminus \cup_k B_k) = 0$ a.e. Hence $\mu_t = \nu_t$ a.e., and $S = \sum_{i=1}^\infty S_i$. The lemma follows since $S_1 + \cdots + S_n$ is locally elementary for each n .

THEOREM 8.7. *Suppose L_p is a p -Banach function space ($0 < p < 1$) satisfying (8.1.1). Suppose L_p embeds in X . Then there is an embedding $S: L_p \rightarrow X$ which is a lattice isomorphism, i.e., $S(f \wedge g) = Sf \wedge Sg$, $f, g \in L_p$ (i.e., L_p embeds as a sublattice).*

PROOF. Suppose L_p does not embed as a sublattice. Let $\mathcal{G}$ be the set of all operators T such that whenever $B \in \mathfrak{B}$ and $\lambda(B) > 0$ then $T|_{L_p(B)}$ fails to be an isomorphism. An application of Proposition 4.2 shows that $\mathcal{G}$ is closed under local

convergence. We prove the theorem by contradiction by showing $\mathcal{G} = \mathcal{L}(L_p, X)$; by Lemma 8.6 we need only show every locally elementary operator is in $\mathcal{G}$. Clearly to do this it suffices to show each elementary operator belongs to $\mathcal{G}$. Suppose S is elementary and for some B , $\lambda(B) > 0$, SP_B is an embedding. Then $|SP_B|$ is a lattice isomorphism and an embedding. This contradicts our hypothesis and so $\mathcal{G} = \mathcal{L}(L_p, X)$ and the theorem follows.

9. Symmetric function spaces. Let us denote by $\mathcal{G}$ the set of all functions $F: [0, \infty) \rightarrow [0, 1]$ which are monotone decreasing right-continuous and such that $\lim_{x \rightarrow \infty} F(x) = 0$. Let Λ_p ($0 < p < \infty$) denote the class of maps $\Phi: \mathcal{G} \rightarrow [0, \infty]$ such that:

(9.0.1) If $F, G \in \mathcal{G}$ and $F \leq G$ then $\Phi(F) \leq \Phi(G)$.

(9.0.2) If $F_n, F \in \mathcal{G}$ and $F_n(x) \uparrow F(x)$, $0 \leq x < \infty$ then $\lim_{n \rightarrow \infty} \Phi(F_n) = \Phi(F)$.

(9.0.3) If $F, G \in \mathcal{G}$ and $F(0) + G(0) \leq 1$, $\Phi(F + G) \leq \Phi(F) + \Phi(G)$.

(9.0.4) If $\theta > 0$ and $F \in \mathcal{G}$ then $\Phi(F_\theta) = \theta^{-p} \Phi(F)$ where $F_\theta(x) = F(\theta x)$, $x > 0$.

(9.0.5) For some $F \in \mathcal{G}$, $0 < \Phi(F) < \infty$.

Note that by (9.0.4), $\Phi(0) = 0$.

Then we define the space $L[p; \Phi]$ to be the space of all $f \in L_0$ such that $\|f\| = \Phi(F)^{1/p} < \infty$ where $F(x) = \lambda(|f| > x)$.

THEOREM 9.1. *If $\Phi \in \Lambda_p$, $L[p; \Phi]$ is a σ -complete quasi-Banach function space having the Fatou property (2.0.7) and such that if $f_1, f_2 \in L[p; \Phi]$ have disjoint supports*

$$\|f_1 + f_2\|^p \leq \|f_1\|^p + \|f_2\|^p. \quad (9.1.1)$$

PROOF. First we observe that (9.0.3), (9.0.4), and (9.0.5) together imply $0 < \Phi(F) < \infty$ for every simple function F . Hence $L[p; \Phi]$ includes all simple functions.

Next observe that (9.1.1) holds by applying (9.0.3). We use this to show that $\|\cdot\|$ is a quasinorm. For suppose $f, g, h \in L[p; \Phi]$ and $h = f + g$. Choose $A \in \mathfrak{B}$ with $\lambda(A) = 1/2$. Then $h1_A = f1_A + g1_A$ and

$$\lambda\left(|h1_A| > \frac{1}{2}x\right) \leq \lambda\left(|f1_A| > \frac{1}{2}x\right) + \lambda\left(|g1_A| > \frac{1}{2}x\right),$$

so that

$$\|h1_A\|^p \leq 2^p(\|f1_A\|^p + \|g1_A\|^p).$$

Similarly,

$$\|h1_{K-A}\|^p \leq 2^p(\|f1_{K-A}\|^p + \|g1_{K-A}\|^p).$$

Hence,

$$\|h\|^p \leq 2^{p+1}(\|f\|^p + \|g\|^p),$$

so that

$$\|h\| \leq 2^{2/p}(\|f\| + \|g\|).$$

Next we show the inclusion $L[p; \Phi] \rightarrow L_0$ is continuous. Indeed if $\|f_n\| \rightarrow 0$ then if $\varepsilon > 0$ and $\delta_n = \lambda(|f_n| > \varepsilon)$, then if we denote $F_n(x) = \lambda(|f_n| > x)$, $F_n > \delta_n 1_{[0, \varepsilon)}$,

and hence, $\Phi(\delta_n 1_{[0, \epsilon]}) \rightarrow 0$. If $\limsup \delta_n > \delta > 0$, we obtain $\Phi(\delta 1_{[0, \epsilon]}) = 0$, contradicting our initial remarks. Hence $f_n \rightarrow 0$ in measure.

Now $\|\cdot\|$ is clearly a symmetric lattice quasinorm with the Fatou property and $L[p; \Phi]$ is a σ -complete lattice. To show $L[p; \Phi]$ is complete it suffices to show that if $f_n > 0$ and $\|f_n\| \leq 2^{-n}$ then $\sum f_n$ converges. Clearly $\sum f_n$ converges in L_0 to g say. If G_n is the distribution of $\sum_{i=1}^n f_i$ and G the distribution of g then $G_n(x) \uparrow G(x)$ for $0 < x < \infty$ and $\sup \Phi(G_n) < \infty$. Hence $\Phi(G) < \infty$ and $g \in L[p; \Phi]$. A similar argument shows that

$$\left\| g - \sum_{i=1}^n f_i \right\| \leq \limsup_{m \rightarrow \infty} \left\| \sum_{i=1}^m f_i - \sum_{i=1}^n f_i \right\| \rightarrow 0 \quad \text{and} \quad n \rightarrow \infty.$$

THEOREM 9.2. *Suppose X is a p -convex symmetric σ -complete quasi-Banach function space. Then $X = L[p; \Phi]$ for some $\Phi \in \Lambda_p$.*

PROOF. We may suppose the lattice quasinorm on X is symmetric, has the Fatou property and satisfies (9.1.1). Then define

$$\begin{aligned} \Phi(F) &= \|f\|^p, & f \in X, \\ &= \infty, & f \notin X, \end{aligned}$$

where $F \in \mathcal{G}$ and $f \in L_0$ is such that $\lambda(|f| > x) = F(x)$, $x > 0$.

We do not know whether every $L[p; \Phi]$ is p -convex. However we obtain a positive result with one further hypothesis.

DEFINITION. $\Phi \in \Lambda_p$ is *totally symmetric* if

$$\Phi(tF) = t\Phi(F), \quad F \in \mathcal{G}, 0 < t \leq 1. \quad (9.2.1)$$

A σ -complete quasi-Banach function space X is *totally symmetric of order p* if $X = L[p; \Phi]$ for some totally symmetric $\Phi \in \Lambda_p$.

In terms of the quasinorm on $L[p; \Phi]$, total symmetry implies that if $B \in \mathfrak{B}$ and $\sigma: B \rightarrow K$ is any Borel map such that $\lambda(\sigma^{-1}A) = \lambda(A)\lambda(B)$, $t \in \mathfrak{B}$, then the map $R_B: L[p; \Phi] \rightarrow L[p; \Phi]$ given by

$$\begin{aligned} R_B f(t) &= f(\sigma t), & t \in B, \\ &= 0, & t \notin B, \end{aligned}$$

satisfies $\|R_B f\| = \lambda(B)^{1/p} \|f\|$, $f \in L[p; \Phi]$.

Of course the spaces $L[p; \infty]$ and L_p are totally symmetric of order p ; so are the intermediate Lorentz space $L(p; q)$, where $p < q < \infty$.

LEMMA 9.3. *Suppose $\Phi \in \Lambda_p$ is totally symmetric and $0 < p < 1$; then there is a countable collection $\mathfrak{M}(\Phi)$ of continuous increasing functions $M: [0, \infty) \rightarrow [0, \infty)$ such that:*

$$M(0) = 0. \quad (9.3.1)$$

$$M(x + y) \leq M(x) + M(y), \quad x, y \geq 0. \quad (9.3.2)$$

$$(1 - p)\Phi(F) \leq \sup_{M \in \mathfrak{M}(\Phi)} \int_0^\infty F(x) dM(x) \leq \Phi(F), \quad f \in \mathcal{G}. \quad (9.3.3)$$

REMARK. $\int_0^\infty F(x) dM(x) = \int_K M(|f(t)|) d\lambda(t)$ if f has distribution F .

PROOF. It clearly suffices to produce $\mathfrak{N}(\Phi)$ so that (9.3.3) holds for simple F . Let $\mathfrak{S}$ be the linear span of the simple functions in $\mathfrak{G}$. For $F \in \mathfrak{S}$ we define

$$\pi(F) = \inf(t\Phi(G): F \leq tG, 0 \leq t < \infty, G \in \mathfrak{G}).$$

Then π is a sublinear functional on $\mathfrak{S}$; indeed it is easy to see that π is positively homogeneous. Further, if $F_i \in \mathfrak{S}$ ($i = 1, 2$) and $\varepsilon > 0$, there exist $G_i \in \mathfrak{G}$ ($i = 1, 2$) and $t_i > 0$ ($i = 1, 2$) such that $t_i\Phi(G_i) \leq \pi(F_i) + \varepsilon/2$ ($i = 1, 2$) and $F_i \leq t_i G_i$ ($i = 1, 2$). Suppose (without loss of generality) that $t_1 > t_2$ so that $t_2 = st_1$ with $0 < s < 1$. Then

$$F_1 + F_2 \leq 2t_1\left(\frac{1}{2}G_1 + \frac{1}{2}sG_2\right)$$

and

$$\pi(F_1 + F_2) \leq 2t_1\left(\Phi\left(\frac{1}{2}G_1\right) + \Phi\left(\frac{1}{2}sG_2\right)\right) \leq \pi(F_1) + \pi(F_2) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, π is sublinear. If $F \leq 0$, $\pi(F) = 0$. If $F \in \mathfrak{G}$ and $F \leq tG$ with $G \in \mathfrak{G}$ and $t > 1$ then $t^{-1}F \leq G$ and hence $\Phi(t^{-1}F) \leq \Phi(G)$. Hence $\pi(F) = \Phi(F)$ for $F \in \mathfrak{G}$.

Now for each $F \in \mathfrak{S}$ with rational values and discontinuities, choose linear $\rho: \delta \rightarrow \mathbb{R}$ such that $\rho(G) \leq \pi(G)$ ($G \in \delta$) and $\rho(F) = \pi(F)$, by the Hahn-Banach Theorem. The collection Δ of all such ρ is countable. If $\rho \in \Delta$, then $\rho(F) > 0$ whenever $F > 0$, since $\pi(F) = 0$ for $F \leq 0$, and $\Phi(F) = \sup_{\rho \in \Delta} \rho(F)$, $F \in \mathfrak{G}$ (since there is a sequence $F_n \in \mathfrak{S}$ with rational values and discontinuities such that $F_n(x) \uparrow F(x)$ for all x).

The function $\theta \mapsto \rho(F_\theta)$ is decreasing if $F \in \mathfrak{G}$ and

$$(1-p) \int_0^1 \rho(F_\theta) d\theta \leq (1-p) \int_0^1 \theta^{-p} \Phi(F) d\theta \leq \Phi(F), \quad (9.3.4)$$

$$(1-p) \int_0^1 \rho(F_\theta) d\theta \geq (1-p)\rho(F), \quad F \in \mathfrak{G} \cap \mathfrak{S}. \quad (9.3.5)$$

For fixed ρ , let

$$N(x) = \rho(1_{[0,x]})$$

and

$$M(x) = (1-p)x \int_x^\infty \frac{N(u)}{u^2} du, \quad x > 0.$$

We let $\mathfrak{N}(\Phi)$ be the set of such functions M . First observe that $M(0) = 0$ and that M is continuous.

Now,

$$(1-p) \int_0^1 \rho(1_{[0,x\theta^{-1}]}) d\theta = (1-p) \int_0^1 N(x\theta^{-1}) d\theta = M(x)$$

upon substituting $u = x\theta^{-1}$. Since ρ is a positive linear functional, M is an increasing function. Further, if $F \in \mathfrak{S} \cap \mathfrak{G}$, $(1-p) \int_0^1 \rho(F_\theta) d\theta = \int_0^\infty F(x) dM(x)$ (since M is continuous). Now (9.3.4) and (9.3.5) imply (9.3.3). To conclude, we show (9.3.2) holds. Indeed,

$$\frac{M(x)}{x} = (1-p) \int_x^\infty \frac{N(u)}{u^2} du$$

is monotone decreasing and this implies (9.3.2).

THEOREM 9.4. *If $0 < p < 1$ and $\Phi \in \Lambda_p$ is totally symmetric, $L[p; \Phi]$ is p -convex.*

PROOF. Suppose $f_1, \dots, f_n \in L[p; \Phi]$ and $h = f_1 + \dots + f_n$. For $\varepsilon > 0$ select $M \in \mathfrak{M}(\Phi)$ so that

$$\int_K M(|h(t)|) d\lambda(t) > (1-p)[\|h\|^p - \varepsilon].$$

Then $(1-p)[\|h\|^p - \varepsilon] \leq \sum_{i=1}^n \int_K M(|f_i(t)|) d\lambda(t)$ by (9.3.2) and hence,

$$(1-p)[\|h\|^p - \varepsilon] \leq \sum_{i=1}^n \|f_i\|^p,$$

for $\varepsilon > 0$ is arbitrary,

$$\|h\|^p \leq (1-p)^{-1} \sum_{i=1}^n \|f_i\|^p.$$

Now let $\mathcal{G}_\infty$ be the set of all monotone decreasing right-continuous functions $F: [0, \infty) \rightarrow [0, \infty]$ such that $\lim_{x \rightarrow \infty} F(x) = 0$. If $\Phi \in \Lambda_p$, we define its *symmetric extension* $\Psi: \mathcal{G}_\infty \rightarrow [0, \infty]$ by

$$\Psi(F) = \sup\{t\Phi(G): G \in \mathcal{G}, t > 0, tG \leq F\}.$$

Then Ψ satisfies the following conditions:

(9.4.1) If $F, G \in \mathcal{G}_\infty$, and $F \leq G$, then $\Psi(F) \leq \Psi(G)$.

(9.4.2) If $F_n, F \in \mathcal{G}_\infty$ and $F_n(x) \uparrow F(x)$, $0 \leq x < \infty$, then $\lim_{n \rightarrow \infty} \Psi(F_n) = \Psi(F)$.

(9.4.3) If $F, G \in \mathcal{G}_\infty$, $\Psi(F + G) \leq \Psi(F) + \Psi(G)$.

(9.4.4) If $\theta > 0$ and $F \in \mathcal{G}_\infty$, $\Psi(F_\theta) = \theta^{-p} \Psi(F)$.

(9.4.5) If $F \in \mathcal{G}_\infty$ and $t > 0$, $\Psi(tF) = t\Psi(F)$.

Thus, provided there exists $F \in \mathcal{G}$ with $\Psi(F) < \infty$, then $\Psi|_{\mathcal{G}} \in \Lambda_p$ and is totally symmetric.

LEMMA 9.5. *For $F \in \mathcal{G}$,*

(9.5.1) $\Psi(F) \geq \Phi(F)$,

(9.5.2) $\Psi(F) \leq \limsup_{t \rightarrow 0} t^{-1} \Phi(tF)$,

(9.5.3) *if Φ is totally symmetric, $\Phi(F) = \Psi(F)$.*

PROOF. Only (9.5.2) requires proof. If $\varepsilon > 0$, $G \in \mathcal{G}$ and $t > 0$ such that $tG \leq F$ and $t\Phi(G) \geq \Psi(F) - \varepsilon/2$. For $\tau = m/n \in \mathcal{Q}$, so that $\tau < t$, $\tau\Phi(G) \geq \Psi(F) - \varepsilon$. Thus for any N ,

$$Nn\left(\frac{G}{Nn}\right) = G$$

and hence,

$$\Phi\left(\frac{G}{Nn}\right) \geq \frac{1}{Nn} \Phi(G).$$

However,

$$\frac{G}{Nn} \leq \frac{F}{Nm}$$

and hence,

$$\Phi\left(\frac{F}{Nm}\right) \geq \frac{1}{Nn} \Phi(G) \geq \frac{1}{Nm} (\Psi(F) - \epsilon).$$

The result now follows.

THEOREM 9.6. *Suppose X is a symmetric σ -complete p -Banach function space, where $0 < p < 1$, so that $X = L[p; \Phi]$ for some $\Phi \in \Lambda_p$. Let Ψ be the symmetric extension of Φ . Suppose $T \in \mathcal{L}(L_p, L_0)$ has distribution $w(t, x)$. Then a necessary and sufficient condition that $T \in \mathcal{L}(L_p, X)$ is that*

$$\operatorname{ess\,sup}_{t \in K} \Psi(w(t, \cdot)) = C < \infty \quad (9.6.1)$$

and then $\|T\| \leq C^{1/p}(1-p)^{-1/p}$.

PROOF. Let ν_t be the kernel of T and $\alpha_x(B) = \int_K \Delta_x(\nu_t)[B] d\lambda(t)$, $B \in \mathfrak{B}$, $x > 0$, so that $\alpha_x(B) = \int_B w(t, x) d\lambda(t)$. Consider the condition

$$\Psi[\alpha_x(B)] \leq C^* \lambda(B), \quad B \in \mathfrak{B}. \quad (9.6.2)$$

If $\Psi(F) = \infty$, unless $F = 0$, (9.6.1) and (9.6.2) are trivially equivalent.

Otherwise, we observe that if (9.6.2) holds, then for $M \in \mathfrak{N}(\Psi)$,

$$\int_0^\infty \int_B w(t, x) d\lambda(t) dM(x) \leq C^* \lambda(B), \quad B \in \mathfrak{B},$$

and so by Fubini's theorem,

$$\int_0^\infty w(t, x) dM(x) \leq C^* \quad \text{a.e.}$$

Since $\mathfrak{N}(\Psi)$ is countable, we obtain

$$\Psi(w(t, \cdot)) \leq C^*(1-p)^{-1} \quad \text{a.e.}$$

Thus (9.6.2) implies (9.6.1) with $C = C^*(1-p)^{-1}$. Reversing the reasoning (9.6.1) implies (9.6.2) with $C^* = C(1-p)^{-1}$.

Now suppose (9.6.2) holds. Then if T is elementary, (9.6.2) implies that $T \in \mathcal{L}(L_p, X)$ and $\|T\| \leq (C^*)^{1/p}$, since $\alpha_x(B)$ is the distribution of $T1_B$. Hence this statement also holds for locally elementary T . To obtain the result for general T , we appeal to Lemma 8.4. It is sufficient to show $|T| \in \mathcal{L}(L_p, X)$, and Lemma 8.4 allows us to define an increasing sequence V_n of positive locally elementary operators such that $V_n f \uparrow Tf$ a.e. for $f \geq 0$. Clearly the preceding argument shows $\|V_n\| \leq (C^*)^{1/p}$ and so $\|T\| \leq (C^*)^{1/p}$.

Conversely, suppose $T \in \mathcal{L}(L_p, X)$. Without loss of generality we may suppose $T \geq 0$. Suppose first T is elementary and of the form $Tf(t) = a(t)f(\sigma t)$ where $a > 0$ is a simple Borel function and $\sigma: K \rightarrow K$ is Borel. Let $b_1, \dots, b_l$ be the nonzero values of a and let $B_i = \{t: a(t) = b_i\}$. Since T is continuous into L_0 , the measures $\gamma_i(E) = \lambda(\sigma^{-1}E \cap B_i)$, $E \in \mathfrak{B}$, are absolutely continuous, and hence we may write $\gamma_i(C) = \int_C h_i(t) d\lambda(t)$, $E \in \mathfrak{B}$, where $h_1, \dots, h_l$ are positive Borel functions.

Next decompose K into Borel sets $A_1, A_2, \dots$ such that $c_{ij} \leq h_i(t) \leq 2c_{ij}$, $t \in A_j$, $i = 1, 2, \dots, l$. Suppose $E \subset A_j$ (some j) and $F \subset E$ are both Borel sets of positive measure. Then the distributions of $T1_F$ and $T1_E$ are $x \mapsto \alpha_x(F)$ and $x \mapsto \alpha_x(E)$ and these must satisfy

$$\frac{1}{2} \frac{\lambda(F)}{\lambda(E)} \alpha_x(E) \leq \alpha_x(F), \quad x \geq 0,$$

and hence,

$$\Phi\left(\frac{1}{2} \frac{\lambda(F)}{\lambda(E)} \alpha_x(E)\right) \leq \|T\|^p \lambda(F), \quad F \subset E.$$

Allowing F to vary we obtain

$$\Psi(\alpha_x(E)) \leq 2\|T\|^p \lambda(E), \quad E \subset A_j. \quad (9.6.3)$$

Since Ψ is subadditive, (9.6.3) holds for all $E \in \mathfrak{B}$.

For general positive elementary operators we simply approximate by an increasing sequence of "simple" elementary operators as above to obtain (9.6.3). This gives (9.6.3) also for locally elementary operators and hence by Lemma 8.4 for all $T \in \mathcal{L}(L_p, X)$.

COROLLARY 9.7. $\mathcal{L}(L_p, X) \neq \{0\}$ if and only if $L_p \subset X$.

PROOF. $\mathcal{L}(L_p, X) \neq \{0\}$ if and only if $\Psi(F) < \infty$ for some $F \neq 0$. If Ψ is totally symmetric and $\Psi(1_{[0,1]}) = 1$, then it is easy to see $\Psi(F) \leq \int_0^\infty F(x) d(x^p)$. Hence $\Psi(F) < \infty$ for some $F \neq 0$ if and only if $\Psi(F) \leq c \int_0^\infty F(x) x^{p-1} dx$ for some c and this is if and only if $\Phi(F) \leq c \int_0^\infty F(x) x^{p-1} dx$, i.e., $L[p; \Phi] \supset L_p$.

THEOREM 9.8. Let X be a separable symmetric σ -complete p -convex function space. Then X is isomorphic to a quotient of L_p if and only if $X = L[p; \Phi]$ for a totally symmetric Φ .

PROOF. Suppose $X = L[p; \Phi]$ where Φ is totally symmetric. Since X is separable the simple functions are dense in X . Suppose f is simple; then f may be "split" so that

$$f = f_1^n + f_2^n + \dots + f_{2^n}^n$$

where $f_{2k-1}^{n+1} + f_{2k}^{n+1} = f_k^{n+1}$ ($1 \leq k \leq 2^n$), then f_k^n ($1 \leq k \leq 2^n$) have disjoint support and identical distributions F_n . If F is the distribution of f , $F_n = 2^{-n}F$.

Now let $E(n; j)$ be the partitioning of K described in Proposition 4.2. We define $T: L_p \rightarrow L[p; \Phi]$ such that $T1_{E(n; j)} = f_j^n$. Then, $\|T1_{E(n; j)}\| \leq 2^{-n/p} \|f\|$ and hence $\|T\| \leq c \|f\|$, where c is a constant such that

$$\|f_1 + \dots + f_n\|^p \leq c^p (\|f_1\|^p + \dots + \|f_n\|^p)$$

for $g_1, \dots, g_n \in L[p; \Phi]$.

Let $\{h_n\}$ be a sequence of simple functions dense in the unit ball of $L[p; \Phi]$. For each n , select as above $T \in \mathcal{L}(L_p, L[p, \Phi])$ so that $\|T\| \leq c$ and $T1_K = h_n$. Define

$$S: l_p(L_p) \rightarrow L[p; \Phi], \quad S(f_n) = \sum T_n f_n.$$

Then $\|S\| < c^2$ and is almost open. Hence S is a surjection; since $l_p(L_p) \cong L_p$, the theorem is proved.

Conversely if X is a quotient of L_p , the pair (L_p, X) is transitive (cf. [12]) i.e. given $f \in X$ there exists $T \in \mathcal{L}(L_p, X)$ such that $T1_K = f$. However, Theorem 9.6 shows that if $X = L[p; \Phi]$ then each $T \in \mathcal{L}(L_p, X)$ maps into $L[p, \Psi]$ where Ψ is the symmetric extension of Φ . Hence, $L[p, \Psi] = L[p, \Phi] = X$.

REMARK. If we remove the condition of σ -completeness we must allow the possibility that X is the closure of the simple functions in some $L[p; \Phi]$ for totally symmetric Φ .

EXAMPLE 9.9. If $p < q < \infty$, $L[p, q]$ is a quotient of L_p which contains no copy of l_p .

PROOF. It is only necessary to show l_p does not embed in $L[p, q]$. If $F_1, \dots, F_n \in \mathcal{G}_\infty$,

$$\left(\int_0^\infty \left(\sum F_i(x) \right)^{q/p} x^{q-1} dx \right)^{p/q} > c \left(\sum \left(\int_0^\infty F_i(x)^{q/p} x^{q-1} dx \right)^{p/q} \right)^{1/r},$$

where $r = 2$ if $q \leq 2p$ and $r = q/p$ if $q > 2p$, and c is some constant (see [25]). Hence if $f_1, \dots, f_n \in L[p, q]$ have disjoint supports,

$$\|f_1 + \dots + f_n\|^p > c \left(\sum \|f_i\|^p \right)^{1/r}$$

so $L(p, q)$ satisfies (8.1.1). Hence if l_p embeds in $L[p, q]$ there is a sequence f_n with disjoint supports so that $\|f_i\| \leq 1$, but for all n and all $l_1 < l_2 < \dots < l_n$,

$$\|f_{l_1} + \dots + f_{l_n}\| > cn^{1/p}$$

where $c > 0$. Suppose $F_1, \dots, F_n$ are the distributions of these functions. Since $\lambda(\text{supp } f_n) \rightarrow 0$, $F_n(x) \rightarrow 0$ uniformly on $[0, \infty)$. By passing to a subsequence we may suppose F_n basic in $L_{q/p}((0, \infty), x^{q-1} dx)$ and equivalent to a basic sequence with disjoint supports. Thus,

$$\left\{ \int_0^\infty \left(\sum_{i=1}^n F_i(x) \right)^{q/p} x^{q-1} dx \right\}^{p/q} < Mn^{p/q}.$$

Thus $cn^{1/p} \leq Mn^{1/q}$ for all n . This contradiction shows that l_p does not embed in $L[p, q]$.

We conclude §9 by giving a partial result on the classification of operators on $L(p, \infty)$ for $0 < p < 1$.

THEOREM 9.10. Suppose $T: L(p, \infty) \rightarrow L(p, \infty)$ is positive. Then $T \in \mathcal{L}(L_p)$ and $\|T\|_p \leq \|T\|_{p, \infty}$.

PROOF. We shall suppose $\|T\|_{p, \infty} \leq 1$. Consider the restriction $T: L_p \rightarrow L(p, \infty)$. We suppose first that T is elementary and (as in Theorem 9.6) $Tf(t) = a(t)f(\sigma t)$, where $a: K \rightarrow \mathbb{R}$ is simple and positive. Suppose $b_1 > b_2 > \dots > b_l > 0$ are the positive values assumed by a and let $B_j = \{t: a(t) = b_j\}$. Let $h_1, \dots, h_l$ be positive Borel functions so that $\lambda(\sigma^{-1}E \cap B_j) = \int_E h_j(t) d\lambda(t)$, $E \in \mathcal{B}$, for $j = 1, 2, \dots, l$.

Suppose $d > 1$ is arbitrary. Then we can decompose K into Borel sets $\{A_i: i = 1, 2, \dots\}$ such that $c_{ij} \leq h_j(t) \leq dc_{ij}$, $t \in A_i$, $j = 1, 2, \dots, l$. Fix i and let

$0 < \varepsilon < \lambda(A_i)$ be such that

$$\varepsilon < \min(b_1^{-1}, b_2^{-1}, \dots, b_l^{-1}, b_1, b_2, \dots, b_l).$$

Let $f \in L_p(A_i)$ be a function whose distribution satisfies

$$\begin{aligned} F(x) &= \varepsilon, & 0 \leq x \leq \varepsilon, \\ &= \varepsilon^{1+p} x^{-p}, & \varepsilon \leq x < \varepsilon^{-1}, \\ &= 0, & x > \varepsilon^{-1}. \end{aligned}$$

Then $\|f\|_{p,\infty} = \sup_{0 < x < \infty} xF(x)^{1/p} = \varepsilon^{1+1/p}$.

Now $\|Tf\|_{p,\infty} \leq \varepsilon^{1+1/p}$, but Tf has distribution G satisfying

$$G(x) \geq \sum_{j=1}^l F(xb_j^{-1})c_{ij}, \quad 0 < x < \infty,$$

and hence

$$G(1) \geq \sum_{j=1}^l F(b_j^{-1})c_{ij} = \sum_{j=1}^l \varepsilon^{1+p} b_j^p c_{ij}.$$

Thus

$$\sum_{j=1}^l \varepsilon^{1+p} b_j^p c_{ij} \leq \varepsilon^{1+p},$$

i.e.,

$$\sum_{j=1}^l b_j^p c_{ij} \leq 1.$$

Now if $E \subset A_i$ is Borel then

$$\begin{aligned} \|T1_E\|^p &= \sum_{j=1}^l b_j^p \lambda(\sigma^{-1}E \cap B_j) = \sum_{j=1}^l b_j^p \int_E h_j(t) d\lambda(t) \\ &\leq d \sum_{j=1}^l b_j^p c_{ij} \lambda(E) \leq d \lambda(E). \end{aligned}$$

Hence $\|TP_{A_i}\|_p \leq d^{1/p}$. As this is true for all i ,

$$\|T\|_p \leq d^{1/p}$$

and as $d > 1$ is arbitrary,

$$\|T\|_p \leq 1. \quad (9.10.1)$$

Now it follows that (9.10.1) holds for elementary T by approximation of $a(t)$. Then (9.10.1) also holds for locally elementary T .

Now we apply Lemma 8.4 to introduce the sequence S_n . Then for each n $0 < S_1 + \dots + S_n \leq T$ and hence $\|S_1 + \dots + S_n\|_p \leq 1$. From this it follows easily that $\|T\|_p \leq 1$.

10. Miscellaneous results. Suppose G is a compact metrizable group and λ is Haar measure on G . Then the translation invariant operators $T: L_p \rightarrow L_p$ ($0 < p < 1$) and $T: L_p \rightarrow L(p, \infty)$ have been classified by Sawyer [30] (for the circle group) and Oberlin [23], [24] for general locally compact groups.

Suppose Φ is a totally symmetric function in Λ_p . Then $l(p; \Phi)$ denotes the sequence space of all (ξ_n) such that $\Phi(F) < \infty$ where $F(x)$ is the number of n such that $|\xi_n| > x$, and we also denote by Φ the symmetric extension of Φ . For simplicity we shall suppose G abelian in the next two theorems.

THEOREM 10.1. *Suppose $T: L_p(G) \rightarrow L[p, \Phi](G)$ ($0 < p < 1$) is translation invariant. Then*

$$Tf = \mu * f, \quad f \in L_p, \quad (10.1.1)$$

where $\mu = \sum_{n=1}^{\infty} \xi_n \delta(g_n)$, with $(g_n: n \in \mathbb{N})$ a sequence of points in G and $(\xi_n) \in l(p; \Phi)$. Conversely, if $(\xi_n) \in l(p; \Phi)$, (10.1.1) defines a translation-invariant operator $T \in \mathcal{L}(L_p, L[p; \Phi])$.

PROOF. Suppose T has kernel $g \mapsto \nu_g$. Then for a dense sequence $\{f_n\} \subset C(G)$, we have, for fixed $g \in G$

$$\begin{aligned} Tf_n(gh) &= \int_G f_n(t) d\nu_{gh} \quad \text{a.e., } h \in G, \\ &= \int_G f_n(gt) d\nu_h \quad \text{a.e., } h \in G. \end{aligned}$$

In particular $\|\nu_g\|$ is a translation-invariant function in $L^\infty(G)$; hence $\|\nu_g\|$ is constant almost everywhere and so $T \in \mathcal{L}(L_1)$. Thus (see [33]), $Tf = \mu * f, f \in L_1$, where $\mu \in \mathfrak{M}(K)$. By the uniqueness of the kernel, $Tf = \mu * f, f \in L_p$, and $\mu = \sum_{n=1}^{\infty} \xi_n \delta(g_n)$ where $g_n \in G$ are distinct, and $\sum |\xi_n| < \infty$. Now $\nu_h = \sum_{n=1}^{\infty} \xi_n \delta(h^{-1}g_n)$ a.e. and hence $\Delta_x(\nu_h) = \Delta_x(\mu)$ a.e. Thus T has distribution w where

$$\begin{aligned} w(t, x) &= \Delta_x(\mu) \quad \text{a.e., } 0 \leq x < \infty, \\ &= F(x), \end{aligned}$$

where $F(x)$ is the number of n such that $|\xi_n| > x$. Then $T \in \mathcal{L}(L_p, L[p; \Phi])$ provided $\Phi(F) < \infty$. The converse is easy.

EXAMPLES. (1) $T \in \mathcal{L}(L_p, L[p, \infty])$ if and only if $\xi_n \in l(p, \infty)$, i.e., $\sup_n n^{1/p} \xi_n^* < \infty$, where (ξ_n^*) is the decreasing rearrangement of (ξ_n) (see Oberlin [24]).

(2) $T \in \mathcal{L}(L_p; L[p; q])$ ($p < q < \infty$) if and only if $\xi_n \in l(p; q)$, i.e.

$$\begin{aligned} q/p \sum_{n=1}^{\infty} (\xi_n^*)^q \int_{n-1}^n t^{q/p-1} dt &< \infty, \\ \sum_{n=1}^{\infty} (\xi_n^*)^q [n^{q/p} - (n-1)^{q/p}] &< \infty \end{aligned}$$

which is equivalent to

$$\sum_{n=1}^{\infty} (\xi_n^*)^q n^{q/p-1} < \infty.$$

(3) If $T \in \mathcal{L}(L_p, L_0)$ is translation invariant, the same proof combined with Nikisin's theorem shows that $\sup_n n^{1/p} \xi_n^* < \infty$, i.e. $T \in \mathcal{L}(L_p, L(p; \infty))$.

THEOREM 10.2. Suppose $\mu \in \mathfrak{M}(G)$ and for each $f \in L[p; \infty]$, $\mu * f$ is well defined, i.e. for almost every $g \in G$,

$$\int_G |f(gh)| d|\mu|(h) < \infty. \quad (10.2.1)$$

Then

$$\mu = \sum \xi_n \delta(g_n) \quad (10.2.2)$$

where (g_n) is a sequence of disjoint points in G and

$$\sum |\xi_n|^p < \infty. \quad (10.2.3)$$

Conversely, if (10.2.2) and (10.2.3) hold then $\mu * f$ is defined for all $f \in L(p, \infty)$.

PROOF. If $\mu * f$ is defined for each $f \in L(p, \infty)$, we define

$$Tf = |\mu| * f, \quad f \in L[p; \infty].$$

It is easy to see (by considering $f \in L_1$ first) that $Tf \in L_0$. Also if $\|f_n\| \leq 2^{-n}$, then $\sum |f_n|$ converges in $L(p; \infty)$. Hence $|\mu| * \sum |f_n|$ is well defined and so $|\mu| * |f_n| \rightarrow 0$ a.e. Then $|\mu| * f_n \rightarrow 0$ a.e. Thus $T: L[p; \infty] \rightarrow L_0$ is a linear operator.

Now by Nikisin's theorem there is a positive function $\varphi \in L_0$ such that $M_\varphi T \in \mathcal{L}(L(p; \infty))$ where

$$M_\varphi f(t) = \varphi(t)f(t).$$

Now $M_\varphi T \geq 0$ and hence by Theorem 9.10, $M_\varphi T \in \mathcal{L}(L_p)$. Now this implies that $\varphi(t)\|\mu\|_p < \infty$ for almost every t ; thus (10.2.2) and (10.2.3) follow.

The converse follows easily from the p -convexity of $L[p; \infty]$.

To conclude, we observe that our main theorem 7.2 implies a similar result for an F -space X in place of L_0 provided there are enough linear operators $T \in \mathcal{L}(X, L_0)$ to separate points. It is easy enough to produce spaces X which fail this property and for which $\mathcal{L}(X, L_0) = \{0\}$. An example is the space L_p/H_p considered in [10] (for the same reasons as in the proof of $\mathcal{L}(L_p/H_p, L_p) = \{0\}$ given in that paper). We now prove a related result concerning this space. For convenience we convert it to a real space, by the process of taking real and imaginary parts.

Let $K = \Gamma_1 \cup \Gamma_2$ where Γ_1 and Γ_2 denote two disjoint copies of the unit circle $\Gamma = \{z: |z| = 1\}$. Let $\lambda = 1/2(\lambda_1 + \lambda_2)$ where λ_1 and λ_2 denote Haar measure on Γ_1 and Γ_2 respectively. Let $\hat{H}_p$ be the closed subspace of $L_p(K)$ generated by the function (where $0 \leq n < \infty$)

$$\begin{aligned} e_n(z) &= \operatorname{Re} z^n, & z \in \Gamma_1, \\ &= \operatorname{Im} z^n, & z \in \Gamma_2, \\ e_n^*(z) &= -\operatorname{Im} z^n, & z \in \Gamma_1, \\ &= \operatorname{Re} z^n, & z \in \Gamma_2. \end{aligned}$$

Let $f \in \hat{H}_p$ if and only if $f_1(z) + if_2(z) \in H_p$ where

$$\begin{aligned} f_1(z) &= f(z), & z \in \Gamma_1, \\ f_2(z) &= f(z), & z \in \Gamma_2. \end{aligned}$$

Thus $\hat{H}_p$ is a proper closed subspace of L_p .

THEOREM 10.3. *Let X be a σ -complete p -Banach lattice with order-continuous quasinorm (i.e. such that for any monotone decreasing sequence x_n with $\inf x_n = 0$ then $\inf \|x_n\| = 0$). Then $\mathcal{L}(L_p/\hat{H}_p, X) = \{0\}$.*

PROOF. Suppose $T \in \mathcal{L}(L_p/\hat{H}_p, X)$, and let $Q: L_p \rightarrow L_p/\hat{H}_p$ be the quotient map. Let $S = |TQ| \in \mathcal{L}(L_p, X)$, and let $S1_K = u \in X$. Let $Y \subset X$ be the linear span of $[-u, u]$ with $[-u, u]$ as its unit ball; then Y is a Banach lattice which is an AM-space [35, p. 22]. Hence Y is isometrically isomorphic to a space $C(\Omega)$ where Ω is a compact Hausdorff space; since Y is order-complete, Ω is Stonian (see [35, pp. 59 and 92]). We shall identify Y and $C(\Omega)$.

Now $T(C(K)) \subset C(\Omega)$ and $T: C(K) \rightarrow C(\Omega)$ is a linear map of norm one at most. Hence

$$Tf(\omega) = \int_K f d\nu_\omega, \quad f \in C(K),$$

where $\nu_\omega \in \mu(K)$ and $\|\nu_\omega\| \leq 1$.

For any $\omega \in \Omega$,

$$\int_K e_n d\nu_\omega = \int_K e_n^* d\nu_\omega = 0, \quad n = 0, 1, 2, \dots$$

Let μ_1 and μ_2 be the measures induced on Γ by $\nu_\omega|_{\Gamma_1}$ and $\nu_\omega|_{\Gamma_2}$. Then

$$\begin{aligned} \int_\Gamma \operatorname{Re} z^n d\mu_1 + \int_\Gamma \operatorname{Im} z^n d\mu_2 &= 0, \\ - \int_\Gamma \operatorname{Im} z^n d\mu_1 + \int_\Gamma \operatorname{Re} z^n d\mu_2 &= 0, \end{aligned}$$

so that

$$\int_\Gamma z^n d(\mu_1 - i\mu_2) = 0.$$

Now by the F. and M. Riesz Theorem [6, p. 41], $\mu_1 - i\mu_2$ is absolutely continuous with respect to Haar measure on the circle. Thus ν_ω is λ -continuous for every $\omega \in \Omega$.

Let $U = \{f \in L_p(K): \|f\|_\infty \leq 1\}$. We shall show $T(U)$ is relatively compact. Suppose not; then we may find a sequence of continuous functions f_n such that

$$\|Tf_n - Tf_m\| \geq \varepsilon > 0, \quad m \neq n,$$

and $\|f_n\|_\infty \leq 1$. By passing to a subsequence we may suppose (f_n) converges in $\sigma(L_\infty, L_1)$ and hence that if $g_n = f_n - f_{n+1}$, $g_n \rightarrow 0$, $\sigma(L_\infty, L_1)$, $\|Tg_n\| \geq \varepsilon$, $n = 1, 2, \dots$. Now $Tg_n(\omega) \rightarrow 0$, $\omega \in \Omega$. Let

$$h_n = \sup(|Tg_n|, |Tg_{n+1}|, \dots)$$

in X . Then $h_n \leq u$ and $h_n \in Y$. Hence

$$h_n(\omega) = \sup_n |Tg_n(\omega)|$$

except on a set of first category in ω . Thus $h_n(\omega) \rightarrow 0$ except on a set of first category. It follows that $\inf h_n = 0$ and so $\inf_n \|h_n\| = 0$. However, $\|h_n\| \geq \sup \|Tg_n\| \geq \varepsilon$.

It follows that $T(U)$ is compact and hence $T(U) = 0$ (see [10]).

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